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Introduction to Particle Accelerator Physics

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Closest *rms* Fit Ellipses

For zero-centered distributions, i.e., distributions that have zero average value for x and x'

$$\varepsilon_{rms} \equiv \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$

$$\beta = \frac{\langle x^2 \rangle}{\varepsilon_{rms}} = \frac{\sigma_x^2}{\varepsilon_{rms}}$$

$$\alpha = -\frac{\langle xx' \rangle}{\varepsilon_{rms}}$$

$$\gamma = \frac{\langle x'^2 \rangle}{\varepsilon_{rms}} = \frac{\sigma_{x'}^2}{\varepsilon_{rms}}$$

Case: Uniformly Filled Ellipse

$$\rho(x,x') = \frac{1}{\pi\varepsilon}\Theta\left(1 - \frac{\gamma x^2 + 2\alpha x x' + \beta x'^2}{\varepsilon}\right)$$

Θ here is the Heavyside step function, 1 for positive values of its argument and zero for negative values for its argument

$$\sigma_x^2 = \left\langle x^2 \right\rangle = \frac{\varepsilon\beta}{4}$$

$$\left\langle x x' \right\rangle = -\frac{\varepsilon\alpha}{4}$$

$$\sigma_{x'}^2 = \left\langle x'^2 \right\rangle = \frac{\varepsilon}{4\beta}\left(1+\alpha^2\right)$$

$$\therefore \varepsilon_{rms} = \frac{\varepsilon}{4}$$

Effects back on particle beam

Classically; radiation causes damping of the transverse and longitudinal (energy or synchrotron) oscillations in the particle beam. For ring accelerators the time scale for the damping of the oscillations is easy to estimate

$$\tau_{damping} \equiv \frac{1}{f_{rev}} \frac{E}{\delta E_{turn}}$$

In more detail, the oscillations will follow a damped sinusoid

$$\begin{pmatrix} x \\ y \\ \Delta\phi \end{pmatrix}(t) = \begin{pmatrix} x_0 \exp\left(-t/\tau_x\right)\cos\left(\omega_x t - \delta_x\right) \\ y_0 \exp\left(-t/\tau_y\right)\cos\left(\omega_y t - \delta_y\right) \\ \Delta\phi_0 \exp\left(-t/\tau_{\Delta\phi}\right)\cos\left(\omega_{\Delta\phi} t - \delta_{\Delta\phi}\right) \end{pmatrix}$$

Damping rates in detail

For vertical (out of bend plane) damping

$$\tau_y \equiv 2\tau_{damping}$$

For horizontal (in bend plane) damping

$$\tau_x \equiv 2\tau_{damping} / \left(1-\vartheta\right)$$

For vertical (out of bend plane) damping

$$\tau_{\Delta\phi} \equiv 2\tau_{damping} / \left(2+\vartheta\right)$$

Radiation Partition

The damping is partitioned into the various degrees of freedom by

$$\vartheta = \frac{\oint \left(D / \rho^3 \right)\left(1 + 2\rho^2 K \right) ds}{\oint ds / \rho^2}$$

By suitable design (i.e., choice of *D, ρ, K*) one can shift greater or less damping between the horizontal and longitudinal degrees of freedom.

Robinson's Theorem

No matter what choice is made for the radiation partition

$$\frac{1}{\tau_x}+\frac{1}{\tau_y}+\frac{1}{\tau_{\Delta\phi}}=\frac{1}{2\tau_{damping}}\left(1-\vartheta+1+2+\vartheta\right)=\frac{2}{\tau_{damping}}$$

Linearized Synchrotron Oscillations

Need to revisit damping of Synchrotron Oscillations, and energy equilibrium including the energy radiated away

$$eV_c \cos\phi_s = W\left(E_d\right)$$

$$\Delta\phi_{l+1} = \Delta\phi_l - \frac{2\pi L\eta_c}{\lambda\beta_z^2 E_l}\Delta E_l$$

$$\Delta E_{l+1} = \Delta E_l - eV_c \sin\phi_s \Delta\phi_l - \frac{dW}{dE}\Delta E_l$$

$$\frac{d\Delta\phi}{dt} \approx \frac{\Delta\phi_{l+1} - \Delta\phi_l}{T_0} = -\frac{2\pi\eta_c}{\lambda p}\Delta E$$

$$\frac{d\Delta E}{dt} \approx \frac{\Delta E_{l+1} - \Delta E_l}{T_0} = \frac{-eV_c \sin\phi_s \Delta\phi}{T_0} - \frac{1}{T_0}\frac{dW}{dE}\Delta E$$

$$\frac{d^2\Delta E}{dt^2} + \frac{1}{T_0}\frac{dW}{dE}\frac{\Delta E}{dt} + \omega_s^2 \Delta E = 0$$

$$\Delta E\left(t\right) \propto e^{i\omega t} \rightarrow \omega_s^2 + 2\alpha_s i\omega - \omega^2 = 0$$

$$\omega = \alpha_s i \pm \sqrt{\omega_s^2 - \alpha_s^2} \approx \pm\omega_s + i\alpha_s$$

Normal modes are damped with a damping given by

$$\alpha_s = \frac{1}{2T_0}\frac{dW}{dE}$$