# Physics 417/517
# Introduction to Particle Accelerator Physics

G. A. Krafft
Jefferson Lab
Jefferson Lab Professor of Physics
Old Dominion University

# Power Emitted Beam Frame

Larmor/Lienard calculation in the beam frame yields

$$\left\langle P^* \right\rangle = \frac{2e^2}{6\pi\varepsilon_0} cK^2 \left( \frac{2\pi}{\lambda^*} \right)^2 \frac{1}{2}$$

Total energy of each photon is $\hbar 2\pi c/\lambda^*$, therefore the total number of photons radiated after one passage of the insertion device

$$N_\gamma = \frac{2\pi}{3} \alpha N K^2$$

# Spectral Distribution in Beam Frame

Begin with power distribution in beam frame: dipole radiation pattern (single harmonic only when $K$<<1; replace $\gamma$* by $\gamma$, $\beta$* by $\beta$)

$$\frac{dP^*}{d\Omega^*}=\frac{e^2c}{32\pi^2\varepsilon_0}k^{*4}a^2\sin^2\Theta^*$$

Number distribution in terms of wave number

$$\frac{dN_\gamma}{d\Omega^*}=\frac{\alpha}{4}NK^2\frac{k_y^{*2}+k_z^{*2}}{k^{*2}}$$

Solid angle transformation

$$d\Omega^*=\frac{d\Omega}{\gamma^2\left(1-\beta\cos\theta\right)^2}$$

Number distribution in beam frame

$$\frac{dN_\gamma}{d\Omega} = \frac{\alpha}{4} NK^2 \frac{\sin^2\theta \sin^2\varphi + \gamma^2 \left(\cos\theta - \beta\right)^2}{\gamma^4 \left(1 - \beta\cos\theta\right)^4}$$

Energy is simply

$$E(\theta) = \hbar \frac{2\pi\beta c}{\lambda_{ID}\left(1 - \beta\cos\theta\right)} \qquad \hat{E}(\theta) = \frac{1}{\left(1 - \beta\cos\theta\right)}$$

Number distribution as a function of normalized lab-frame energy

$$\frac{dN_\gamma}{d\hat{E}} = \frac{\alpha\pi}{4\gamma^2\beta^3} NK^2 \left[\left(\frac{\hat{E}}{\gamma^2} - 1\right)^2 + \beta^2\right]$$

# Average Energy

Limits of integration

$$\cos\theta = 1 \qquad \hat{E} = \frac{1}{1-\beta} \qquad \cos\theta = -1 \qquad \hat{E} = \frac{1}{1+\beta}$$

Average energy is also analytically calculable

$$\langle E \rangle = \frac{\int_0^\infty E \frac{dN_\gamma}{d\hat{E}} d\hat{E}}{\int_0^\infty \frac{dN_\gamma}{d\hat{E}} d\hat{E}} = \gamma^2 \hbar 2\pi\beta c / \lambda_{ID} \approx \frac{E_{\max}}{2}$$

# Beam *rms* Emittance

Treat the distribution of particles within the beam statistically. Define single particle distribution function

$$\rho\left(x, x'\right)$$

And statistical averaging

$$\left\langle q \right\rangle \equiv \int q\left(x, x'\right) \rho\left(x, x'\right) dx dx'$$

# Closest *rms* Fit Ellipses

For zero-centered distributions, i.e., distributions that have zero average value for $x$ and $x'$

$$\varepsilon_{rms} \equiv \sqrt{\left\langle x^2 \right\rangle \left\langle x'^2 \right\rangle - \left\langle x x' \right\rangle^2}$$

$$\beta = \frac{\left\langle x^2 \right\rangle}{\varepsilon_{rms}} = \frac{\sigma_x^2}{\varepsilon_{rms}}$$

$$\alpha = -\frac{\left\langle x x' \right\rangle}{\varepsilon_{rms}}$$

$$\gamma = \frac{\left\langle x'^2 \right\rangle}{\varepsilon_{rms}} = \frac{\sigma_{x'}^2}{\varepsilon_{rms}}$$

# Case: Uniformly Filled Ellipse

$$\rho(x,x') = \frac{1}{\pi\varepsilon}\Theta\left(1-\frac{\gamma x^2 + 2\alpha x x' + \beta x'^2}{\varepsilon}\right)$$

$\Theta$ here is the Heavyside step function, 1 for positive values of its argument and zero for negative values for its argument

$$\sigma_x^2 = \langle x^2 \rangle = \frac{\varepsilon\beta}{4}$$

$$\langle x x' \rangle = -\frac{\varepsilon\alpha}{4}$$

$$\sigma_{x'}^2 = \langle x'^2 \rangle = \frac{\varepsilon}{4\beta}\left(1+\alpha^2\right)$$

$$\therefore \varepsilon_{rms} = \frac{\varepsilon}{4}$$