Physics 417/517 Introduction to Particle Accelerator Physics

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Solution to Hill's Equation in Amplitude-Phase Form

To get a more general expression for the phase advance, consider in more detail the single particle solutions to Hill's equation

$$\frac{d^2x}{ds^2} + K(s)x = 0$$

From the theory of linear ODEs, the general solution of Hill's equation can be written as the sum of the two linearly independent pseudo-harmonic functions

$$x(s) = Ax_+(s) + Bx_-(s)$$

where

$$x_\pm(s) = w(s)e^{\pm i\mu(s)}$$

are two particular solutions to Hill's equation, provided that

$$\frac{d^2 w}{ds^2} + K(s)w = \frac{c^2}{w^3} \qquad \text{and} \qquad \frac{d\mu}{ds}(s) = \frac{c}{w^2(s)}, \quad \text{Eqns. (3)}$$

and where A, B, and c are constants (in s)

That specific solution with boundary conditions $x(s_1) = x_1$ and $dx/ds\ (s_1) = x'_1$ has

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} w(s_1)e^{i\mu(s_1)} & w(s_1)e^{-i\mu(s_1)} \\ \left[w'(s_1) + \frac{ic}{w(s_1)} \right] e^{i\mu(s_1)} & \left[w'(s_1) - \frac{ic}{w(s_1)} \right] e^{-i\mu(s_1)} \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}$$

Therefore, the unimodular transfer matrix taking the solution at $s = s_1$ to its coordinates at $s = s_2$ is

$$\begin{pmatrix} x_2 \\ x'_2 \end{pmatrix} = \begin{pmatrix} \frac{w(s_2)}{w(s_1)}\cos\Delta\mu_{s_2,s_1} - \frac{w(s_2)w'(s_1)}{c}\sin\Delta\mu_{s_2,s_1} & \frac{w(s_2)w(s_1)}{c}\sin\Delta\mu_{s_2,s_1} \\ -\frac{c}{w(s_2)w(s_1)}\left[1+\frac{w(s_2)w'(s_2)w(s_1)w'(s_1)}{c^2}\right]\sin\Delta\mu_{s_2,s_1} - \left[\frac{w'(s_1)}{w(s_2)} - \frac{w'(s_2)}{w(s_1)}\right]\cos\Delta\mu_{s_2,s_1} & \frac{w(s_1)}{w(s_2)}\cos\Delta\mu_{s_2,s_1} + \frac{w'(s_2)w(s_1)}{c}\sin\Delta\mu_{s_2,s_1} \end{pmatrix} \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}$$

where

$$\Delta\mu_{s_2,s_1} = \mu(s_2) - \mu(s_1) = \int_{s_1}^{s_2} \frac{c}{w^2(s)} ds$$

Case I: $K(s)$ periodic in s

Such boundary conditions, which may be used to describe circular or ring-like accelerators, or periodic focusing lattices, have $K(s + L) = K(s)$. L is either the machine circumference or period length of the focusing lattice.

It is natural to assume that there exists a unique periodic solution $w(s)$ to Eqn. (3a) when $K(s)$ is periodic. Here, we will assume this to be the case. Later, it will be shown how to construct the function explicitly. Clearly for w periodic

$$\phi(s) = \mu(s) - \mu_L s \qquad \text{with} \qquad \mu_L = \int_s^{s+L} \frac{c}{w^2(s)} ds$$

is also periodic by Eqn. (3b), and μ_L is independent of s.

The transfer matrix for a single period reduces to

$$\begin{pmatrix} \cos\mu_L - \dfrac{w(s)w'(s)}{c}\sin\mu_L & \dfrac{w^2(s)}{c}\sin\mu_L \\ -\dfrac{c}{w^2(s)}\left[1+\dfrac{w(s)w'(s)w(s)w'(s)}{c^2}\right]\sin\mu_L & \cos\mu_L + \dfrac{w'(s)w(s)}{c}\sin\mu_L \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(\mu_L) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}\sin(\mu_L)$$

where the (now periodic!) matrix functions are

$$\alpha(s) = -\frac{w(s)w'(s)}{c}, \quad \beta(s) = \frac{w^2(s)}{c}, \quad \gamma(s) = \frac{1+\alpha^2(s)}{\beta(s)}$$

By Thm. (2), these are the ellipse parameters of the periodically repeating, i.e., *matched* ellipses.

General formula for phase advance

In terms of the β-function, the phase advance for the period is

$$\mu_L = \int_0^L \frac{ds}{\beta(s)}$$

and more generally the phase advance between any two longitudinal locations s and s' is

$$\Delta\mu_{s',s} = \int_s^{s'} \frac{ds}{\beta(s)}$$

Transfer Matrix in terms of α and β

Also, the unimodular transfer matrix taking the solution from s to s' is

$$M_{s',s} = \begin{pmatrix} \sqrt{\dfrac{\beta(s')}{\beta(s)}}\left(\cos\Delta\mu_{s',s} + \alpha(s)\sin\Delta\mu_{s',s}\right) & \sqrt{\beta(s')\beta(s)}\sin\Delta\mu_{s',s} \\ -\dfrac{1}{\sqrt{\beta(s')\beta(s)}}\begin{bmatrix}(1+\alpha(s')\alpha(s))\sin\Delta\mu_{s',s} \\ +(\alpha(s')-\alpha(s))\cos\Delta\mu_{s',s}\end{bmatrix} & \sqrt{\dfrac{\beta(s)}{\beta(s')}}\left(\cos\Delta\mu_{s',s} - \alpha(s')\sin\Delta\mu_{s',s}\right) \end{pmatrix}$$

Note that this final transfer matrix and the final expression for the phase advance do not depend on the constant c. This conclusion might have been anticipated because different particular solutions to Hill's equation exist for all values of c, but from the theory of linear ordinary differential equations, the final motion is unique once x and dx/ds are specified somewhere.

Method to compute the β-function

Our previous work has indicated a method to compute the β-function (and thus w) directly, i.e., without solving the differential equation Eqn. (3). At a given location s, determine the one-period transfer map $M_{s+L,s}(s)$. From this find μ_L (which is independent of the location chosen!) from $\cos \mu_L = (M_{11}+M_{22}) / 2$, and by choosing the sign of μ_L so that $\beta(s) = M_{12}(s) / \sin \mu_L$ is positive. Likewise, $\alpha(s) = (M_{11}-M_{22}) / 2 \sin \mu_L$. Repeat this exercise at every location the β-function is desired.

By construction, the beta-function and the alpha-function, and hence w, are periodic because the single-period transfer map is periodic. It is straightforward to show $w=(c\beta(s))^{1/2}$ satisfies the envelope equation.

Courant-Snyder Invariant

Consider now a single particular solution of the equations of motion generated by Hill's equation. We've seen that once a particle is on an invariant ellipse for a period, it must stay on that ellipse throughout its motion. Because the phase space area of the single period invariant ellipse is preserved by the motion, the quantity that gives the phase space area of the invariant ellipse in terms of the single particle orbit must also be an invariant. This phase space area/π,

$$\varepsilon = \gamma x^2 + 2\alpha x x' + \beta x'^2 = \left(x^2 + \left(\beta x' + \alpha x\right)^2\right) / \beta$$

is called the Courant-Snyder invariant. It may be verified to be a constant by showing its derivative with respect to s is zero by Hill's equation, or by explicit substitution of the transfer matrix solution which begins at some initial value $s = 0$.

Pseudoharmonic Solution

$$\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta_0}}\left(\cos\Delta\mu_{s,0} + \alpha_0 \sin\Delta\mu_{s,0}\right) & \sqrt{\beta(s)\beta_0}\sin\Delta\mu_{s,0} \\ -\frac{1}{\sqrt{\beta(s)\beta_0}}\begin{bmatrix}(1+\alpha(s)\alpha_0)\sin\Delta\mu_{s,0} \\ +(\alpha(s)-\alpha_0)\cos\Delta\mu_{s,0}\end{bmatrix} & \sqrt{\frac{\beta_0}{\beta(s)}}\left(\cos\Delta\mu_{s,0} - \alpha(s)\sin\Delta\mu_{s,0}\right) \end{pmatrix} \begin{pmatrix} x_0 \\ \left.\frac{dx}{ds}\right|_0 \end{pmatrix}$$

gives

$$\left(x^2(s) + \left(\beta(s)x'(s) + \alpha(s)x(s)\right)^2\right)/\beta(s) = \left(x_0^2 + \left(\beta_0 x'_0 + \alpha_0 x_0\right)^2\right)/\beta_0 \equiv \varepsilon$$

Using the $x(s)$ equation above and the definition of ε, the solution may be written in the standard "pseudoharmonic" form

$$x(s) = \sqrt{\varepsilon\beta(s)}\cos\left(\Delta\mu_{s,0} - \delta\right) \quad \text{where } \delta = \tan^{-1}\left(\frac{\beta_0 x'_0 + \alpha_0 x_0}{x_0}\right)$$

The the origin of the terminology "phase advance" is now obvious.

Dispersion Calculation

Begin with the inhomogeneous Hill's equation for the dispersion.

$$\frac{d^2D}{ds^2} + K(s)D = \frac{1}{\rho(s)}$$

Write the general solution to the inhomogeneous equation for the dispersion as before.

$$D(s) = D_p(s) + Ax_1(s) + Bx_2(s)$$

Here D_p can be any particular solution. Suppose that the dispersion and it's derivative are known at the location s_1, and we wish to determine their values at s_2. x_1 and x_2, because they are solutions to the homogeneous equations, must be transported by the transfer matrix solution $M_{s2,s1}$ already found.

To build up the general solution, choose that particular solution of the inhomogeneous equation with boundary conditions

$$D_{p,0}\left(s_1\right) = D'_{p,0}\left(s_1\right) = 0$$

Evaluate A and B by the requirement that the dispersion and it's derivative have the proper value at s_1 (x_1 and x_2 need to be linearly independent!)

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} x_1\left(s_1\right) & x_2\left(s_1\right) \\ x'_1\left(s_1\right) & x'_2\left(s_1\right) \end{pmatrix}^{-1} \begin{pmatrix} D\left(s_1\right) \\ D'\left(s_1\right) \end{pmatrix}$$

$$D\left(s_2\right) = D_{p,0}\left(s_2 - s_1\right) + \left(M_{s_2,s_1}\right)_{11} D\left(s_1\right) + \left(M_{s_2,s_1}\right)_{12} D'\left(s_1\right)$$

$$D'\left(s_2\right) = D'_{p,0}\left(s_2 - s_1\right) + \left(M_{s_2,s_1}\right)_{21} D\left(s_1\right) + \left(M_{s_2,s_1}\right)_{22} D'\left(s_1\right)$$

3 by 3 Matrices for Dispersion Tracking

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} \left(M_{s_2,s_1}\right)_{11} & \left(M_{s_2,s_1}\right)_{12} & D_{p,0}(s_2 - s_1) \\ \left(M_{s_2,s_1}\right)_{21} & \left(M_{s_2,s_1}\right)_{22} & D'_{p,0}(s_2 - s_1) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}$$

Particular solutions to inhomogeneous equation for constant K and constant ρ and vanishing dispersion and derivative at $s = 0$

	$K < 0$	$K = 0$	$K > 0$
$D_{p,0}(s)$	$\frac{1}{\lvert K\rvert\rho}\left(\cosh\left(\sqrt{\lvert K\rvert}s\right)-1\right)$	$\frac{s^2}{2\rho}$	$\frac{1}{K\rho}\left(1-\cos\left(\sqrt{K}s\right)\right)$
$D'_{p,0}(s)$	$\frac{1}{\sqrt{\lvert K\rvert}\rho}\sinh\left(\sqrt{\lvert K\rvert}s\right)$	$\frac{s}{\rho}$	$\frac{1}{\sqrt{K}\rho}\sin\left(\sqrt{K}s\right)$