Physics 417/517 Introduction to Particle Accelerator Physics

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Phase Advance of a Unimodular Matrix

Any two-by-two unimodular (Det (M) = 1) matrix with |Tr M| < 2 can be written in the form

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(\mu) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin(\mu)$$

The *phase advance* of the matrix, μ, gives the eigenvalues of the matrix $\lambda = e^{\pm i\mu}$, and $\cos \mu = (\mathrm{Tr}\, M)/2$. Furthermore $\beta\gamma - \alpha^2 = 1$

Pf: The equation for the eigenvalues of M is

$$\lambda^2 - (M_{11} + M_{22})\lambda + 1 = 0$$

Because M is real, both λ and λ^* are solutions of the quadratic. Because

$$\lambda = \frac{\mathrm{Tr}(M)}{2} \pm i\sqrt{1-(\mathrm{Tr}(M)/2)^2}$$

For $|\mathrm{Tr}\, M| < 2$, $\lambda\, \lambda^* = 1$ and so $\lambda_{1,2} = e^{\pm i\mu}$. Consequently $\cos\mu = (\mathrm{Tr}\, M)/2$. Now the following matrix is trace-free.

$$M - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(\mu) = \begin{pmatrix} \dfrac{M_{11} - M_{22}}{2} & M_{12} \\ M_{21} & \dfrac{M_{22} - M_{11}}{2} \end{pmatrix}$$

Simply choose

$$\alpha = \frac{M_{11} - M_{22}}{2\sin\mu}, \qquad \beta = \frac{M_{12}}{\sin\mu}, \qquad \gamma = -\frac{M_{21}}{\sin\mu}$$

and the sign of μ to properly match the individual matrix elements with $\beta > 0$. It is easily verified that $\beta\gamma - \alpha^2 = 1$. Now

$$M^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(2\mu) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}\sin(2\mu)$$

and more generally

$$M^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(n\mu) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}\sin(n\mu)$$

Therefore, because sin and cos are both bounded functions, the matrix elements of any power of M remain bounded as long as $|\mathrm{Tr}\,(M)| < 2$.

NB, in some beam dynamics literature it is (incorrectly!) stated that the less stringent $|\mathrm{Tr}\,(M)| \leq 2$ ensures boundedness and/or stability. That equality cannot be allowed can be immediately demonstrated by counterexample. The upper triangular or lower triangular subgroups of the two-by-two unimodular matrices, i.e., matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$$

clearly have unbounded powers if $|x|$ is not equal to 0.

Significance of matrix parameters

Another way to interpret the parameters α, β, and γ, which represent the unimodular matrix M (these parameters are sometimes called the Twiss parameters or Twiss representation for the matrix) is as the "coordinates" of that specific set of ellipses that are mapped onto each other, or are invariant, under the linear action of the matrix. This result is demonstrated in

Thm: For the unimodular linear transformation

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(\mu) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin(\mu)$$

with $|\mathrm{Tr}\,(M)| < 2$, the ellipses

$$\gamma x^2 + 2\alpha xy + \beta y^2 = c$$

are invariant under the linear action of M, where c is any constant. Furthermore, these are the only invariant ellipses. Note that the theorem does not apply to $\pm I$, because $|\mathrm{Tr}\,(\pm I)| = 2$.

Pf: The inverse to M is clearly

$$M^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(\mu) - \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin(\mu)$$

By the ellipse transformation formulas, for example

$$\beta' = \beta^2 \left(\sin^2 \mu\right)\gamma + 2\left(-\beta \sin \mu\right)\left(\cos \mu + \alpha \sin \mu\right)\alpha + \left(\cos \mu + \alpha \sin \mu\right)^2 \beta$$

$$= \beta \sin^2 \mu \left(1 + \alpha^2\right) - 2\beta\alpha^2 \sin^2 \mu + \beta \cos^2 \mu + \beta\alpha^2 \sin^2 \mu$$

$$= \left(\sin^2 \mu + \cos^2 \mu\right)\beta = \beta$$

Similar calculations demonstrate that $\alpha' = \alpha$ and $\gamma' = \gamma$. As det $(M) = 1$, $c' = c$, and therefore the ellipse is invariant. Conversely, suppose that an ellipse is invariant. By the ellipse transformation formula, the specific ellipse

$$\gamma_i x^2 + 2\alpha_i xy + \beta_i y^2 = \varepsilon$$

is invariant under the transformation by M only if

$$\begin{pmatrix} \gamma_i \\ \alpha_i \\ \beta_i \end{pmatrix} = \begin{pmatrix} (\cos\mu - \alpha\sin\mu)^2 & 2(\cos\mu - \alpha\sin\mu)(\gamma\sin\mu) & (\gamma\sin\mu)^2 \\ -(\cos\mu - \alpha\sin\mu)(\beta\sin\mu) & 1 - 2\beta\gamma\sin^2\mu & (\cos\mu + \alpha\sin\mu)(\gamma\sin\mu) \\ (\beta\sin\mu)^2 & -2(\cos\mu + \alpha\sin\mu)(\beta\sin\mu) & (\cos\mu + \alpha\sin\mu)^2 \end{pmatrix} \begin{pmatrix} \gamma_i \\ \alpha_i \\ \beta_i \end{pmatrix}$$

$$\equiv T_M \begin{pmatrix} \gamma_i \\ \alpha_i \\ \beta_i \end{pmatrix} \equiv T_M \vec{v},$$

i.e., if the vector $\vec{v}$ is *ANY* eigenvector of T_M with eigenvalue 1. All possible solutions may be obtained by investigating the eigenvalues and eigenvectors of T_M. Now

$$T_M \vec{v}_\lambda = \lambda \vec{v}_\lambda \quad \text{has a solution when Det}\left(T_M - \lambda I\right) = 0$$

i.e.,

$$\left(\lambda^2 + \left[2 - 4\cos^2\mu\right]\lambda + 1\right)\left(1 - \lambda\right) = 0$$

Therefore, M generates a transformation matrix T_M with at least one eigenvalue equal to 1. For there to be more than one solution with $\lambda = 1$,

$$1 + \left[2 - 4\cos^2\mu\right] + 1 = 0, \quad \cos^2\mu = 1, \quad \text{or} \;\; M = \pm I$$

and we note that *all* ellipses are invariant when $M = \pm I$. But, these two cases are excluded by hypothesis. Therefore, M generates a transformation matrix T_M which always possesses a single nondegenerate eigenvalue 1; the set of eigenvectors corresponding to the eigenvalue 1, all proportional to each other, are the only vectors whose components (γ_i, α_i, β_i) yield equations for the invariant ellipses. For concreteness, compute that eigenvector with eigenvalue 1 normalized so $\beta_i\gamma_i - \alpha_i^2 = 1$

$$\vec{v}_{1,i} = \begin{pmatrix} \gamma_i \\ \alpha_i \\ \beta_i \end{pmatrix} = \beta \begin{pmatrix} -M_{21}/M_{12} \\ \left(M_{11} - M_{22}\right)/2M_{12} \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma \\ \alpha \\ \beta \end{pmatrix}$$

All other eigenvectors with eigenvalue 1 have $\vec{v}_1 = \varepsilon \vec{v}_{1,i} / c$, for some value c.

Because Det (M) =1, the eigenvector $\vec{v}_{1,i}$ clearly yields the invariant ellipse

$$\gamma x^2 + 2\alpha xy + \beta y^2 = \varepsilon.$$

Likewise, the proportional eigenvector $\vec{v}_1$ generates the similar ellipse

$$\frac{\varepsilon}{c}\left(\gamma x^2 + 2\alpha xy + \beta y^2\right) = \varepsilon$$

Because we have enumerated all possible eigenvectors with eigenvalue 1, all ellipses invariant under the action of M, are of the form

$$\gamma x^2 + 2\alpha xy + \beta y^2 = c$$

To summarize, this theorem gives a way to tie the mathematical representation of a unimodular matrix in terms of its α, β, and γ, and its phase advance, to the equations of the ellipses invariant under the matrix transformation. The equations of the invariant ellipses when properly normalized have precisely the same α, β, and γ as in the Twiss representation of the matrix, but varying c.

Finally note that throughout this calculation c acts merely as a scale parameter for the ellipse. All ellipses similar to the starting ellipse, i.e., ellipses whose equations have the same α, β, and γ, but with different c, are also invariant under the action of M. Later, it will be shown that more generally

$$\varepsilon = \gamma x^2 + 2\alpha x x' + \beta x'^2 = \left(x^2 + \left(\beta x' + \alpha x\right)^2\right) / \beta$$

is an invariant of the equations of transverse motion.