Mid Term Examination
Physics 417/517
Due October 11, 2007

1. CERN's Large Hadron Collider (LHC) accelerates protons to very high energy. In the design document the design energy is 7 TeV, or 7000 GeV.
   a. What is the relativistic γ of the accelerated protons? What is their relativistic β?
   $$\gamma = \frac{7000\text{ GeV}}{0.938272\text{ GeV}} = 7460.5$$
   $$\beta = \sqrt{1-1/\gamma^2} = \sqrt{1-1.797\times10^{-8}} = 0.999999991$$
   b. What is their magnetic rigidity?
   $$B\rho = \frac{p}{e} = \frac{\sqrt{\gamma^2-1}\ 938.272\times10^6\text{ V}}{2.998\times10^8\text{ m/sec}} = 23348.9\text{ T m}$$
   c. There are 1232 dipole magnets of length 14.3 m, bending in normal configuration. What is the required dipole magnetic field?
   $$B = \frac{B\rho}{L}2\sin(\theta/2) = \frac{23348.9\text{ T m}}{14.3\text{ m}}2\sin(\pi/1232) = 8.33\text{ T}$$
   d. What is the bend radius when the particles are in the dipoles? Is your result consistent with the total reported machine circumference of 26658.883 m? Explain.
   $$\rho = \frac{B\rho}{B} = \frac{23348.9\text{ T m}}{8.33\text{ T}} = 2803.9\text{ m}$$
   $$\mathrm{C_{bend}} = 2\pi\rho = 17617.6\text{ m} < 26658.883\text{ m}$$
   The difference is made up by the straights containing the quadrupoles, sextupoles, experimental equipment and detectors, etc.
   e. What is the revolution frequency?
   $$f_{rev} = \frac{\beta c}{C} \doteq \frac{2.99792\times10^8\text{ m/sec}}{26658.883\text{ m}} = 11.2455\text{ kHz}$$
   f. The RF frequency is reported to be 400.8 MHz. What is the harmonic number to four significant digits?
   $$h = \frac{f_{RF}}{f_{rev}} = \frac{400800\text{ kHz}}{11.2455\text{ kHz}} = 35641$$
   The reported harmonic number of 35640 MUST be due to a rounding imprecision in the values of the RF frequency in the document.
   g. The total energy gain per pass is 8 MeV. How long (seconds) does it take to accelerate from 450 GeV to 7 TeV?
   $$n_{rev} = \frac{7000\text{ GeV} - 450\text{ GeV}}{0.008\text{ GeV}} = 818750 \quad \Delta t = \frac{818750}{11.2455\times10^3\text{ sec}^{-1}} = 72.8\text{ sec}$$

2. In colliders, to increase the event rate, one would like the beam size very small at the collision point. This leads to the notion of a "low beta insertion". A (grossly!) simplified model of such an insertion is a strong thin focusing lens of focal length f followed by a drift. Suppose $s=0$ at the location of the lens, $\beta(s=0)=\beta_0$ is given, and the ellipse is upright (what does this imply about $\alpha(s=0)$?).

   a. What is $\gamma(s=0)$ for the ellipse?

$$\gamma_0 = \frac{1-0^2}{\beta_0} = \frac{1}{\beta_0}$$

   b. What is the transfer matrix to a location of distance s downstream of the focusing lens?

$$M(0\to s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} = \begin{pmatrix} 1-s/f & s \\ -1/f & 1 \end{pmatrix}$$

   c. What is the inverse of the matrix?

$$M^{-1}(0\to s) = \begin{pmatrix} 1 & -s \\ 1/f & 1-s/f \end{pmatrix}$$

   d. What is $\beta(s)$ (get this from the ellipse transformation formula)?

$$\beta(s) = \left(M^{-1}\right)_{12}^2 \gamma_0 + 0 + \left(M^{-1}\right)_{22}^2 \beta_0 = \frac{s^2}{\beta_0} + \left(1-s/f\right)^2 \beta_0$$

   e. Show $\beta(s)$ is an extremum at location

$$\frac{\beta_0^2}{f\left(1+\beta_0^2/f^2\right)}.$$

$$\frac{d\beta}{ds}\left(s_{extremum}\right) = 0$$

$$\frac{2s_{extremum}}{\beta_0} - \frac{2\left(1-s_{extremum}/f\right)}{f}\beta_0 = 0$$

$$s_{extremum} = \frac{\beta_0^2}{f\left(1+\beta_0^2/f^2\right)}$$

   f. Is it a maximum or minimum?

$$\frac{d^2\beta}{ds^2}\left(s_{extremum}\right) = \frac{2}{\beta_0} + \frac{2\beta_0}{f^2} > 0 \quad \therefore \text{minimum}$$

   g. What is $\beta^* \equiv \beta\left(s_{extremum}\right)$, the value at the location of the extremum?

$$\beta^* = \frac{s_{extremum}^2}{\beta_0} + \left(1 - s_{extremum}/f\right)^2 \beta_0$$

$$= \frac{\beta_0^3}{f^2\left(1+\beta_0^2/f^2\right)^2} + \left(\frac{1+\beta_0^2/f^2 - \beta_0^2/f^2}{1+\beta_0^2/f^2}\right)^2 \beta_0$$

$$= \frac{\beta_0}{1+\beta_0^2/f^2}$$

h. Show the result from part d can be written in the form (Wille, Eqn. 3.151)

$$\beta^* + \frac{\left(s - s_{extremum}\right)^2}{\beta^*}.$$

$$\beta(s) = \frac{s^2}{\beta_0} + \left(1 - s/f\right)^2 \beta_0 = \frac{s^2}{\beta_0} + \left(1 - 2s/f + s^2/f^2\right)\beta_0$$

$$= \frac{s^2}{\beta_0}\left(1+\beta_0^2/f^2\right) - \frac{2s\beta_0}{f} + \beta_0 = \frac{\left(s - s_{extremum}\right)^2}{\beta^*} - \frac{s_{extremum}^2}{\beta^*} + \beta_0$$

$$= \frac{\left(s - s_{extremum}\right)^2}{\beta^*} + \beta^*$$

i. What is $\alpha\left(s_{extremum}\right)$, calculated by the ellipse transformation formula?

$$\alpha\left(s_{extremum}\right) = \left(M^{-1}\right)_{11}\left(M^{-1}\right)_{12}/\beta_0 + \left(M^{-1}\right)_{21}\left(M^{-1}\right)_{22}\beta_0$$

$$= -s_{extremum}/\beta_0 + \left(1/f\right)\left(1 - s_{extremum}/f\right)\beta_0 = \beta_0/f - \frac{s_{extremum}}{\beta_0}\left(1+\beta_0^2/f^2\right) = 0$$

j. Extra Credit: Why does $\alpha\left(s_{extremum}\right)$ have this value? At the extremum $\beta'(s) = -2\alpha(s) = 0$. More physically, before the minimum spot size, also called the beam waste, the beam is converging and after achieving the minimum it is diverging. Therefore, alpha for the ellipse evolves from a positive value to a negative value. At the beam waste it must vanish.

3. Suppose a one period 2 by 2 transfer matrix has $\left|\mathrm{Tr}(M)\right| > 2$.

a. Find an expression for the two eigenvalues in terms of $\mathrm{Tr}(M)$. Call them λ_+ and λ_-.

$$\lambda^2 - \left(M_{11} + M_{12}\right)\lambda + 1 = 0$$

$$\lambda_\pm = \frac{\mathrm{Tr}(M)}{2} \pm \sqrt{\left(\frac{\mathrm{Tr}(M)}{2}\right)^2 - 1}$$

b. Are the two eigenvalues real, or do they have some imaginary components? For $\left|\mathrm{Tr}(M)\right| > 2$, it is clear that the eigenvalues are now purely real because the discriminate of the quadratic is positive.

c. Show $\lambda_+\lambda_- = 1$.

$$\lambda_+\lambda_- = \left(\frac{\mathrm{Tr}(M)}{2} - \sqrt{\left(\frac{\mathrm{Tr}(M)}{2}\right)^2 - 1}\right)\left(\frac{\mathrm{Tr}(M)}{2} - \sqrt{\left(\frac{\mathrm{Tr}(M)}{2}\right)^2 - 1}\right)$$

$$= \left(\frac{\mathrm{Tr}(M)}{2}\right)^2 - \left(\left(\frac{\mathrm{Tr}(M)}{2}\right)^2 - 1\right) = 1$$

Because the eigenvalues are inverses of one another, the absolute value of one eigenvalue is bigger than one, and the absolute value of the other eigenvalue is less than one.

d. By linear algebra, the matrix of the eigenvectors, S, diagonalizes the matrix M

$$\begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} = SMS^{-1}$$

Give an expression for M^n in terms of this diagonal matrix and S.

$$M = S^{-1}\begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}S \qquad M^n = S^{-1}\begin{pmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{pmatrix}S$$

e. Is M bounded as *n* increases? Because one of the eigenvalues has absolute value greater than 1, that eigenvalue raised to larger and larger powers increases without bound. Because *S* is a constant matrix, M^n is not bounded as *n* increase.

f. Comment on the stability or lack of stability of solutions given by repeated application of the transfer matrix M. Repeated application of such a transfer matrix will eventually lead to a displacement from the design orbit as large as one wishes when the initial displacement has some amplitude in the unstable eigenvector. Eventually the displacement is large enough to hit an aperture in the accelerator. So the situation is unstable.

g. Using $\det(AB) = \det(A)\det(B)$, the formula for the determinant of the inverse of a matrix, and part d, comment on the property of the transport matrix that ensures part c is true.

$$\mathrm{Det}\begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} = \lambda_+\lambda_- = \mathrm{Det}(S)\,\mathrm{Det}(M)\,\mathrm{Det}(S^{-1})$$

$$= \mathrm{Det}(S)\,\mathrm{Det}(S^{-1})\,\mathrm{Det}(M) = \mathrm{Det}(M) = 1$$

The unimodular property of *M* ensures that part c is true. Indeed, it applies even in the usual stable case.

h. Show in general (most straightforwardly by completing the matrix multiplication) that

$$\mathrm{Tr}(SMS^{-1}) = \mathrm{Tr}(M)$$

as long as S is an invertible matrix.

Several students pointed out that

$$\mathrm{Tr}(AB)=\mathrm{Tr}(BA)\rightarrow \mathrm{Tr}\left(SMS^{-1}\right)=\mathrm{Tr}\left(SS^{-1}M\right)=\mathrm{Tr}(M)$$

You could also just verify

$$SMS^{-1}=\begin{pmatrix} S_{11}M_{11}+S_{12}M_{21} & S_{11}M_{12}+S_{12}M_{22} \\ S_{21}M_{11}+S_{22}M_{21} & S_{21}M_{12}+S_{22}M_{22} \end{pmatrix}\frac{1}{\mathrm{Det}(S)}\begin{pmatrix} S_{22} & -S_{12} \\ -S_{21} & S_{11} \end{pmatrix}$$

$$=\frac{1}{\mathrm{Det}(S)}\begin{pmatrix} S_{11}S_{22}M_{11}+S_{12}S_{22}M_{21}-S_{11}S_{21}M_{12}-S_{12}S_{21}M_{22} & \cdots \\ \cdots & -S_{21}S_{12}M_{11}-S_{22}S_{12}M_{21}+S_{21}S_{11}M_{12}+S_{22}S_{11}M_{22} \end{pmatrix}$$

$$\mathrm{Tr}\left(SMS^{-1}\right)=\frac{1}{\mathrm{Det}(S)}\left(\mathrm{Det}(S)M_{11}+\mathrm{Det}(S)M_{22}\right)=\mathrm{Tr}(M)$$

4. Recall our expressions from the homework for the transfer matrix of a FODO system. The one-period transfer matrix starting with the middle of the focusing magnet was

$$M_f=\begin{pmatrix} 1 & 0 \\ -1/(2f) & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/(2f) & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 1-L^2/\left(2f^2\right) & 2L+L^2/f \\ -L/\left(2f^2\right)+L^2/\left(4f^3\right) & 1-L^2/\left(2f^2\right) \end{pmatrix}.$$

and M_d is obtained from M_f by replacing f with $-f$

a. How should one choose f, in terms of L, so that the phase advance through one period of the FODO system is 60 degrees (corresponding to 1/6 of a transverse oscillation per period)?

$$\cos\mu=\cos 60^\circ=1/2=\frac{2\left(1-L^2/\left(2f^2\right)\right)}{2}$$

$$\frac{L^2}{f^2}=1\Rightarrow f=L$$

Take the positive solution because we want M_f to be single period matrix starting at the focusing magnet.

b. For the matched phase space (x,x') ellipse in the 60 degree phase advance system, what is the beta-function in the middle of the focusing lens?

$$M_f=\begin{pmatrix} 1/2 & 3L \\ -1/(4L) & 1/2 \end{pmatrix}$$

$$\beta_f=\frac{3L}{\sin 60^\circ}=\frac{3L}{\sqrt{3}/2}=2\sqrt{3}L$$

c. For the matched phase space ellipse in the 60 degree phase advance system, what is the beta-function in the middle of the defocusing lens?

$$M_d = \begin{pmatrix} 1/2 & L \\ -3/(4L) & 1/2 \end{pmatrix}$$

$$\beta_d = \frac{L}{\sin 60^\circ} = \frac{L}{\sqrt{3}/2} = \frac{2L}{\sqrt{3}}$$

d. What are the alpha-functions for the matched ellipses at these same two locations?

$$\alpha = \frac{M_{11} - M_{22}}{2\sin\mu} = 0$$

for both locations.

e. Suppose L is 2 m, and the area of a matched phase space ellipse is $\pi\varepsilon = 10\pi \times 10^{-6}$ m radian, what are the maximum extents ($x_{\max}$) of the matched ellipses in the focusing and defocusing lenses of the 60 degrees phase advance system?

$$x_{f,\max} = \sqrt{\beta_f \varepsilon} = \sqrt{4\sqrt{3}\ \text{m} \cdot 10^{-5}\ \text{m}} = 8.3\ \text{mm}$$

$$x_{d,\max} = \sqrt{\beta_d \varepsilon} = \sqrt{\left(4/\sqrt{3}\right)\text{m} \cdot 10^{-5}\ \text{m}} = 4.8\ \text{mm}$$

5. In this problem you are asked to show that for constant uniform focusing, with solutions to Hill's equation given by constant frequency sinusoids, the notion of matching is simplified.

a. Show the transfer matrix from s to s' for a region with uniform focusing strength k is

$$M(s', s) = \begin{pmatrix} \cos\sqrt{k}(s'-s) & \sin\sqrt{k}(s'-s)/\sqrt{k} \\ -\sqrt{k}\sin\sqrt{k}(s'-s) & \cos\sqrt{k}(s'-s) \end{pmatrix}.$$

General Solution Hill's Equation is

$$x(s) = A\cos\sqrt{k}s + B\sin\sqrt{k}s$$

$$x'(s) = -A\sqrt{k}\sin\sqrt{k}s + B\sqrt{k}\cos\sqrt{k}s$$

Initial Condition at s

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{\sqrt{k}} \begin{pmatrix} \sqrt{k}\cos\sqrt{k}s & -\sin\sqrt{k}s \\ \sqrt{k}\sin\sqrt{k}s & \cos\sqrt{k}s \end{pmatrix} \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix}$$

$$\begin{pmatrix} x(s') \\ x'(s') \end{pmatrix} = \begin{pmatrix} \cos\sqrt{k}s' & +\sin\sqrt{k}s' \\ -\sqrt{k}\sin\sqrt{k}s' & \sqrt{k}\cos\sqrt{k}s' \end{pmatrix} \frac{1}{\sqrt{k}} \begin{pmatrix} \sqrt{k}\cos\sqrt{k}s & -\sin\sqrt{k}s \\ \sqrt{k}\sin\sqrt{k}s & \cos\sqrt{k}s \end{pmatrix} \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\sqrt{k}(s'-s) & \sin\sqrt{k}(s'-s)/\sqrt{k} \\ -\sqrt{k}\sin\sqrt{k}(s'-s) & \cos\sqrt{k}(s'-s) \end{pmatrix} \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix}$$

b. What is the phase advance of this matrix?

$$\cos\mu = \frac{\mathrm{Tr}(M)}{2} = \cos\left(\sqrt{k}\,(s'-s)\right)$$

$$\mu = \sqrt{k}\,(s'-s)$$

c. What are the ellipse parameters $\alpha(s), \beta(s), \gamma(s)$ of the matched phase-space ellipses?

$$M_{11} = M_{22} \rightarrow \alpha = 0$$

$$\beta = \frac{M_{12}}{\sin\mu} = \frac{\sin\sqrt{k}\,(s'-s)}{\sqrt{k}\sin\sqrt{k}\,(s'-s)} = \frac{1}{\sqrt{k}}$$

$$\gamma = -\frac{M_{21}}{\sin\mu} = -\frac{-\sqrt{k}\sin\sqrt{k}\,(s'-s)}{\sin\sqrt{k}\,(s'-s)} = \sqrt{k}$$

d. Given a matched phase ellipse entering the region, what are $\alpha(s'), \beta(s'), \gamma(s')$? (Hint: Use the transformation formula.)

$$\gamma(s') = \left(M^{-1}\right)_{11}^2 \sqrt{k} + \left(M^{-1}\right)_{21}^2 \frac{1}{\sqrt{k}} = \sqrt{k}\left(\cos^2\sqrt{k}\,(s'-s) + \sin^2\sqrt{k}\,(s'-s)\right) = \sqrt{k}$$

$$\alpha(s') = \left(M^{-1}\right)_{11}\left(M^{-1}\right)_{12}\sqrt{k} + \left(M^{-1}\right)_{21}\left(M^{-1}\right)_{22}\frac{1}{\sqrt{k}} =$$

$$-\cos\sqrt{k}\,(s'-s)\sin\sqrt{k}\,(s'-s) + \sin\sqrt{k}\,(s'-s)\cos\sqrt{k}\,(s'-s) = 0$$

$$\beta(s') = \left(M^{-1}\right)_{12}^2 \sqrt{k} + \left(M^{-1}\right)_{22}^2 \frac{1}{\sqrt{k}} = \frac{\sin^2\sqrt{k}\,(s'-s) + \cos^2\sqrt{k}\,(s'-s)}{\sqrt{k}} = \frac{1}{\sqrt{k}}$$

e. Interpret. Because the ellipse parameters are independent of s', the matched ellipse does not change as it propagates down the uniform focusing channel. There is just simple phase space rotation without ellipse dimension changes.

f. Suppose $\beta(s) = \beta_{matched}(s) + \Delta\beta$ is slightly above the matched value found in part c, what is $\beta(s')$?

$$\beta(s') = \left(M^{-1}\right)_{12}^2 \sqrt{k} + \left(M^{-1}\right)_{22}^2 \left(\frac{1}{\sqrt{k}} + \Delta\beta\right)$$

$$= \frac{\sin^2\sqrt{k}\,(s'-s) + \cos^2\sqrt{k}\,(s'-s)}{\sqrt{k}} + \Delta\beta\cos^2\sqrt{k}\,(s'-s)$$

$$= \frac{1}{\sqrt{k}} + \frac{\Delta\beta}{2}\left(1 + \cos 2\sqrt{k}\,(s'-s)\right)$$

g. Interpret. The initial deviation in the β from the matched condition is rotated into a γ deviation after ¼ of an oscillation and back to the original ellipse dimensions after ½ of an oscillation. Remember though, that the individual points in phase space are undergoing simple "circular" motions

in phase space. Such so-called mismatch oscillations, can lead to beam quality being degraded.