Physics 417/517
Homework 5 Solution

1. There are many ways to get this answer. Many students just applied Eq. 2.29 of Wille. To get the result from information in the notes, recall that the total energy radiated per turn per electron is

$$\delta E = \frac{4\pi r_e}{3\rho}\beta^3\gamma^3 E_{beam}$$

If the revolution time is T_0 (this quantity will disappear from the final formula!), the power per electron is

$$P_e = \frac{4\pi r_e}{3\rho T_0}\beta^3\gamma^3 E_{beam}\,.$$

But the number of electrons in the ring, in terms of the current, is $I = N_e e / T_0$. So

$$P_{tot} = N_e P_e = \frac{IT_0}{e}\frac{4\pi r_e}{3\rho T_0}\beta^3\gamma^3 E_{beam} = \frac{4\pi r_e}{3\rho}\beta^3\gamma^3 P_{beam}$$

$$\approx \frac{4\pi\cdot 2.82\times 10^{-13}\ \text{cm}}{3\cdot 3896\ \text{cm}}\left(\frac{7000}{0.511}\right)^3 700\ \text{MW} \ = \ 0.546\ \text{MW}.$$

2. From the discussion of undulators, every electron emits

$$\delta E = N_\gamma \left\langle E_\gamma \right\rangle = \left(\frac{2\pi\alpha}{3}NK^2\right)\left(\hbar\frac{2\pi\beta c}{\lambda_{ID}}\gamma^2\right)$$

of energy in photons every turn by each undulator. Following an argument entirely as in problem 1

$$P_u = N_e P_e = \frac{IT_0}{e}\left(\frac{2\pi\alpha}{3}NK^2\right)\left(\hbar\frac{2\pi\beta c}{\lambda_{ID}}\gamma^2\right)\frac{E_{beam}}{T_0\gamma mc^2} = \frac{4\pi^2 r_e}{3\lambda_{ID}}NK^2\beta\gamma P_{beam}$$

$$\approx \frac{4\pi^2\cdot 2.82\times 10^{-13}\cdot 72\cdot 1^2}{3\cdot 2.3}\left(\frac{7000}{0.511}\right)700\ \text{MW} \ = \ 1.114\ \text{kW}.$$

So 50 undulators, which yield about 55.7 kW, are an additional 10% load compared to the bends. So in problem 3, it should be legitimate to neglect undulator contributions compared to the bend contributions to the radiation damping time. A technical point, because $K \sim 1$, there is additional radiation in the harmonics that will raise the total undulator power somewhat.

3. The fundamental damping time obtained from the bend contributions is

$$\tau_{damping} = T_0\frac{E_{beam}}{\delta E} \approx T_0\frac{3\rho}{4\pi r_e\gamma^3} = \frac{1}{271554\ \text{sec}^{-1}}\frac{3\cdot 3896}{4\pi\cdot 2.82\times 10^{-13}}\left(\frac{.511}{7000}\right)^3 = 4.725\ \text{msec}$$

With the natural partition, because $\vartheta \ll 1$,

$$\tau_x \approx\ 9.45\ \text{msec},\ \ \tau_y \approx\ 9.45\ \text{msec},\ \text{and}\ t_{\Delta\phi} \approx\ 4.725\ \text{msec}.$$

These results agree very well with the values in the report.

4. This may have been "the worst" problem of the semester, as it required the largest leap beyond what was presented in class. The following argument is taken from Sands, *Physics of Electron Storage Rings*, the justifiably famous SLAC-121 report referenced in Wille. Hisham found the report and the link was distributed on the 29th. My argument will follow Sands, but use the notation we used during the semester. Start with the amplitude growth formula from class

$$\left\langle \frac{dA^2}{dt} \right\rangle = \int_0^\infty \Delta e^2 \frac{d\dot{n}}{d(\Delta e)} d(\Delta e),$$

where Δe is the energy of the emitted photons. Note there are contributions only for those times when the particles are in the bends emitting. The integral we evaluate in the now-standard way

$$\int_0^\infty \Delta e^2 \frac{d\dot{n}}{d(\Delta e)} d(\Delta e) = \frac{\sqrt{3}}{8\pi^2\varepsilon_0}\frac{e^2}{\rho}\frac{\gamma}{\hbar\omega_c}\omega_c\hbar^2\omega_c^2\int_0^\infty \xi^2 \int_\xi^\infty K_{5/3}(x)dxd\xi$$

$$= \frac{\sqrt{3}}{8\pi^2\varepsilon_0}\frac{e^2\gamma}{\rho}\frac{\hbar\omega_c^2}{3}\frac{55\pi}{27} = \frac{\sqrt{3}}{32\pi^2\varepsilon_0}\frac{e^2\hbar c^2\gamma^7}{\rho^3}\frac{55\pi}{9}.$$

Note the ρ^{-3} dependence, and the strong energy dependence of this result. Because $ds \approx cdt$, for one turn the total change in the amplitude is

$$\Delta\left\langle A^2 \right\rangle = \frac{\sqrt{3}}{32\pi^2\varepsilon_0} e^2\hbar c\gamma^7 \frac{55\pi}{9}\oint \frac{ds}{\rho^3}.$$

The excitation rate averaged over the full circumference of the ring is therefore

$$\frac{d\left\langle A^2 \right\rangle}{dt} = \frac{\sqrt{3}}{32\pi^2\varepsilon_0}\frac{e^2\hbar c^2\gamma^7}{L}\frac{55\pi}{9}\oint\frac{ds}{\rho^3} \equiv \frac{\sqrt{3}}{32\pi^2\varepsilon_0} e^2\hbar c^2\gamma^7 \frac{55\pi}{9}\left\langle \frac{1}{\rho^3} \right\rangle,$$

where the ring average is defined by

$$\left\langle \frac{1}{\rho^3} \right\rangle \equiv \oint \frac{ds}{\rho^3(s)} / \oint ds = \oint \frac{ds}{\rho^3(s)} / L.$$

As in the lectures, the amplitude damping is (make sure you understand the 2)

$$\frac{d\left\langle A^2 \right\rangle}{dt} = -\frac{2}{\tau_{\Delta\phi}}\left\langle A^2 \right\rangle$$

Equilibrium requires

$$\left\langle A^2 \right\rangle = \frac{\tau_{\Delta\phi}}{2}\frac{\sqrt{3}}{32\pi^2\varepsilon_0} e^2\hbar c^2\gamma^7 \frac{55\pi}{9}\left\langle \frac{1}{\rho^3} \right\rangle = \frac{\frac{2L}{2}\frac{\sqrt{3}}{32\pi^2\varepsilon_0} e^2\hbar c\gamma^7 \frac{55\pi}{9}\left\langle \frac{1}{\rho^3} \right\rangle}{(2+\vartheta)\delta E/E}.$$

As is usual for bound oscillations in equilibrium, the amplitude is shared equally between the coordinate and the conjugate variable. Therefore $\sigma_E^2 = \left\langle A^2 \right\rangle / 2$.

$$\frac{\left\langle A^2 \right\rangle}{E^2} = \frac{2\sigma_E^2}{E^2} = \frac{\frac{2L}{2}\frac{\sqrt{3}}{32\pi^2\varepsilon_0}\frac{e^2\hbar c\gamma^6}{mc^2}\frac{55\pi}{9}\left\langle \frac{1}{\rho^3} \right\rangle}{(2+\vartheta)\delta E}$$

$$\delta E \approx \frac{e^2}{3\varepsilon_0 \rho}\gamma^4 = \frac{e^2\gamma^4}{6\pi\varepsilon_0}\oint \frac{ds}{\rho^2} \to \frac{\sigma_E^2}{E^2} = \frac{\frac{1}{32}\frac{\hbar c\gamma^2}{mc^2(2+\vartheta)}\frac{55}{\sqrt{3}}\left\langle \frac{1}{\rho^3}\right\rangle}{\left\langle \frac{1}{\rho^2}\right\rangle}$$

5. The first task to do is the integrals in the hint:

$$\int_{1/(1+\beta)}^{1/(1-\beta)} d\hat{E} = \hat{E}\Big|_{1/(1+\beta)}^{1/(1-\beta)} = \gamma^2\left(1+\beta-(1-\beta)\right) = 2\beta\gamma^2$$

$$\int_{1/(1+\beta)}^{1/(1-\beta)} \hat{E}d\hat{E} = \frac{\hat{E}^2}{2}\Bigg|_{1/(1+\beta)}^{1/(1-\beta)} = \frac{\gamma^4}{2}\left(1^2+2\beta+\beta^2-\left(1^2-2\beta+\beta^2\right)\right) = 2\beta\gamma^4$$

$$\int_{1/(1+\beta)}^{1/(1-\beta)} \hat{E}^2 d\hat{E} = \frac{\hat{E}^3}{3}\Bigg|_{1/(1+\beta)}^{1/(1-\beta)} = \frac{\gamma^6}{3}\left(1+3\beta+3\beta^2+\beta^3-\left(1-3\beta+3\beta^2-\beta^3\right)\right)$$

$$= \frac{\gamma^6}{3}\left(6\beta+2\beta^3\right)$$

$$\int_{1/(1+\beta)}^{1/(1-\beta)} \hat{E}^3 d\hat{E} = \frac{\hat{E}^4}{4}\Bigg|_{1/(1+\beta)}^{1/(1-\beta)} = \frac{\gamma^8}{4}\begin{pmatrix} 1+4\beta+6\beta^2+4\beta^3+\beta^4 \\ -\left(1-4\beta+6\beta^2-4\beta^3+\beta^4\right)\end{pmatrix}$$

$$= 2\gamma^8\left(\beta+\beta^3\right)$$

$$\int_{1/(1+\beta)}^{1/(1-\beta)} \hat{E}^4 d\hat{E} = \frac{\hat{E}^5}{5}\Bigg|_{1/(1+\beta)}^{1/(1-\beta)} = \frac{\gamma^{10}}{5}\begin{pmatrix} 1+5\beta+10\beta^2+10\beta^3+5\beta^4+\beta^5 \\ -\left(1-5\beta+10\beta^2-10\beta^3+5\beta^4-\beta^5\right)\end{pmatrix}$$

$$= \frac{\gamma^{10}}{5}\left(10\beta+20\beta^3+2\beta^5\right)$$

Next evaluate the relevant integrals (for $K \ll 1, \quad \beta \approx \beta^* \quad \gamma \approx \gamma^*$)

$$N_\gamma = \int_{1/(1+\beta)}^{1/(1-\beta)} \frac{dN_\gamma}{d\hat{E}} d\hat{E} = \frac{\alpha\pi NK^2}{4\gamma^2\beta^3}\int_{1/(1+\beta)}^{1/(1-\beta)}\left[\left(\frac{\hat{E}}{\gamma^2}-1\right)^2+\beta^2\right]d\hat{E}$$

$$\frac{\alpha\pi NK^2}{4\gamma^2\beta^3}\int_{1/(1+\beta)}^{1/(1-\beta)}\left[\frac{\hat{E}^2}{\gamma^4}-2\frac{\hat{E}}{\gamma^2}+1+\beta^2\right]d\hat{E}$$

$$= \frac{\alpha\pi NK^2}{4\gamma^2\beta^3}\left[\frac{\gamma^6\left(6\beta+2\beta^3\right)}{3\gamma^4}-2\frac{2\beta\gamma^4}{\gamma^2}+\left(1+\beta^2\right)2\beta\gamma^2\right]$$

$$= \frac{\alpha\pi NK^2}{4\gamma^2\beta^3}\frac{8}{3}\beta^3\gamma^2 = \frac{2\pi\alpha NK^2}{3}$$

$$\left\langle E_\gamma \right\rangle = \frac{\int_{1/(1+\beta)}^{1/(1-\beta)} E_\gamma \frac{dN_\gamma}{d\hat{E}} d\hat{E}}{\int_{1/(1+\beta)}^{1/(1-\beta)} \frac{dN_\gamma}{d\hat{E}} d\hat{E}} = \frac{\frac{\hbar 2\pi\beta c}{\lambda_{ID}} \int_{1/(1+\beta)}^{1/(1-\beta)} \hat{E}\left[\left(\frac{\hat{E}}{\gamma^2}-1\right)^2+\beta^2\right] d\hat{E}}{8\gamma^2\beta^3/3}$$

$$= \frac{\frac{\hbar 2\pi\beta c}{\lambda_{ID}} \int_{1/(1+\beta)}^{1/(1-\beta)} \left[\frac{\hat{E}^3}{\gamma^4} - 2\frac{\hat{E}^2}{\gamma^2} + \left(1+\beta^2\right)\hat{E}\right] d\hat{E}}{8\gamma^2\beta^3/3}$$

$$= \frac{\frac{\hbar 2\pi\beta c}{\lambda_{ID}} \left[\frac{2\gamma^8\left(\beta+\beta^3\right)}{\gamma^4} - 2\frac{\gamma^6\left(6\beta+2\beta^3\right)}{3\gamma^2} + \left(1+\beta^2\right)2\beta\gamma^4\right]}{8\gamma^2\beta^3/3}$$

$$= \frac{\frac{\hbar 2\pi\beta c}{\lambda_{ID}} \gamma^4\beta^3\left[2-\frac{4}{3}+2\right]}{8\gamma^2\beta^3/3} = \frac{\hbar 2\pi\beta c}{\lambda_{ID}}\gamma^2$$

$$\left\langle E_\gamma^2 \right\rangle = \frac{\int_{1/(1+\beta)}^{1/(1-\beta)} E_\gamma^2 \frac{dN_\gamma}{d\hat{E}} d\hat{E}}{\int_{1/(1+\beta)}^{1/(1-\beta)} \frac{dN_\gamma}{d\hat{E}} d\hat{E}} = \frac{\left(\frac{\hbar 2\pi\beta c}{\lambda_{ID}}\right)^2 \int_{1/(1+\beta)}^{1/(1-\beta)} \hat{E}^2\left[\left(\frac{\hat{E}}{\gamma^2}-1\right)^2+\beta^2\right] d\hat{E}}{8\gamma^2\beta^3/3}$$

$$= \frac{\left(\frac{\hbar 2\pi\beta c}{\lambda_{ID}}\right)^2 \int_{1/(1+\beta)}^{1/(1-\beta)} \left[\frac{\hat{E}^4}{\gamma^4} - 2\frac{\hat{E}^3}{\gamma^2} + \left(1+\beta^2\right)\hat{E}^2\right] d\hat{E}}{8\gamma^2\beta^3/3}$$

$$= \frac{\left(\frac{\hbar 2\pi\beta c}{\lambda_{ID}}\right)^2 \left[\frac{\gamma^{10}\left(10\beta+20\beta^3+2\beta^5\right)}{5\gamma^4} - 2\frac{2\gamma^8\left(\beta+\beta^3\right)}{\gamma^2} + \left(1+\beta^2\right)\gamma^6\left(6\beta+2\beta^3\right)/3\right]}{8\gamma^2\beta^3/3}$$

$$= \frac{\left(\frac{\hbar 2\pi\beta c}{\lambda_{ID}}\right)^2 \left[\frac{\gamma^{10}\left(10\beta+20\beta^3+2\beta^5\right)}{5\gamma^4} - 2\frac{2\gamma^8\left(\beta+\beta^3\right)}{\gamma^2} + \left(1+\beta^2\right)\gamma^6\left(6\beta+2\beta^3\right)/3\right]}{8\gamma^2\beta^3/3}$$

$$= \frac{\left(\frac{\hbar 2\pi\beta c}{\lambda_{ID}}\right)^2 \gamma^6\left[\left(2-4+2\right)\beta + \left(4-4+2+2/3\right)\beta^3 + \left(2/5+2/3\right)\beta^5\right]}{8\gamma^2\beta^3/3} = \left(\frac{\hbar 2\pi\beta c}{\lambda_{ID}}\right)^2 \gamma^4\left(1+2\beta^2/5\right)$$

$$\therefore E_{\gamma,rms} = \sqrt{\left\langle E_\gamma^2\right\rangle - \left\langle E_\gamma\right\rangle^2} = \frac{\hbar 2\pi\beta c}{\lambda_{ID}}\gamma^2\beta\sqrt{\frac{2}{5}}$$