Physics 417/517
Homework 4 Solutions

1.

$$\omega_c = \frac{3}{2}\frac{2.998\times10^{8}\ \text{m/sec}}{38.96\ \text{m}}\left(\frac{7000}{.511}\right)^{3} = 2.967\times10^{19}\ \text{sec}^{-1}$$

$$\lambda = \frac{(2\pi)2.998\times10^{8}\ \text{m/sec}}{2.967\times10^{19}\ \text{sec}^{-1}} = 6.349\times10^{-11}\ \text{m} = 0.6349\ \text{Angstrom}$$

2.

$$\lambda = \frac{2.3\ \text{cm}}{2(7000/0.511)^{2}}\left(1+K^{2}/2\right)$$

$$= 0.6159\ \text{Angstrom} \quad \text{for } K = 0.1$$

$$= 1.839\ \text{Angstrom} \quad \text{for } K = 2$$

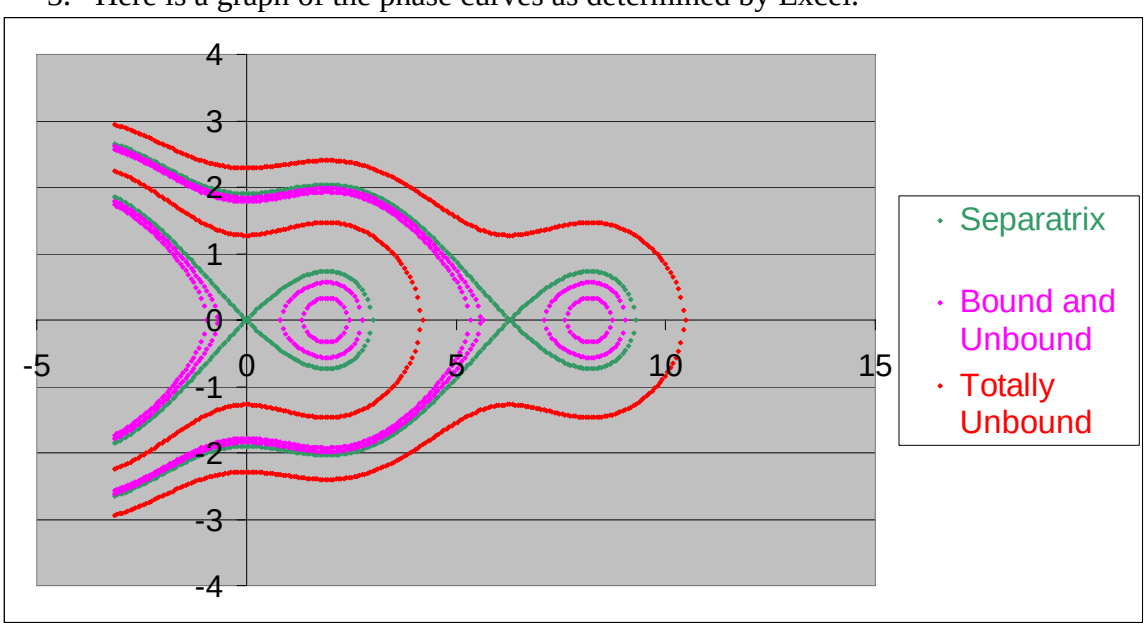
$$\lambda = 2.998\times10^{-7}\ \text{m}$$

$$\gamma^{2} = \frac{2.3\ \text{cm}\cdot1.5}{2\cdot2.998\times10^{-5}\ \text{cm}} = 57538$$

$$E_{beam} = 122.6\ \text{MeV}$$

Unfortunately, there was a mistake in the problem statement. One THz is actually 10^{12} sec^{-1}. Therefore, the number I really wanted was 3.877 MeV!!

3. Here is a graph of the phase curves as determined by Excel.



The abscissa is $\Delta\psi$ is radians, and I've chosen energy units so that the co-efficient in front of the potential term is 1. The green curves are the separatrices of two separate accelerating "buckets", the purple curves are constant-Hamiltonian curves in the "bound" regions (note, at the same value of the Hamiltonian, there are also non-bound orbits), and the red curves are values of the Hamiltonian that only have non-bound orbits. The corresponding curves are separated by $\Delta H = 2\pi \sin\left(35^\circ\right)$.

4. From $d\tau = dt/\gamma$, and the expression for the Lorentz-invariant power

$$P = -\frac{q^2}{6\pi\varepsilon_0 c}\frac{du^\mu}{d\tau}\frac{du_\mu}{d\tau} = -\frac{q^2\gamma^2}{6\pi\varepsilon_0 c}\left[\begin{array}{c} \frac{d\gamma}{dt}\frac{d\gamma}{dt} - \frac{d\left(\gamma\beta_x\right)}{dt}\frac{d\left(\gamma\beta_x\right)}{dt} \\ -\frac{d\left(\gamma\beta_y\right)}{dt}\frac{d\left(\gamma\beta_y\right)}{dt} - \frac{d\left(\gamma\beta_z\right)}{dt}\frac{d\left(\gamma\beta_z\right)}{dt} \end{array}\right]$$

$$P = -\frac{q^2\gamma^2}{6\pi\varepsilon_0 c}\left[\begin{array}{c} \left(1-\beta^2\right)\frac{d\gamma}{dt}\frac{d\gamma}{dt} \\ -2\gamma\frac{d\gamma}{dt}\vec{\beta}\cdot\dot{\vec{\beta}} - \gamma^2\dot{\vec{\beta}}\cdot\dot{\vec{\beta}} \end{array}\right]$$

$$\frac{d\gamma}{dt} = \frac{d}{dt}\left[\frac{1}{\sqrt{1-\beta_x^2-\beta_y^2-\beta_z^2}}\right] = \gamma^3\vec{\beta}\cdot\dot{\vec{\beta}}$$

$$\therefore P = -\frac{q^2\gamma^2}{6\pi\varepsilon_0 c}\left[\begin{array}{c} \gamma^4\left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2 \\ -2\gamma^4\left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2 - \gamma^2\dot{\vec{\beta}}\cdot\dot{\vec{\beta}} \end{array}\right]$$

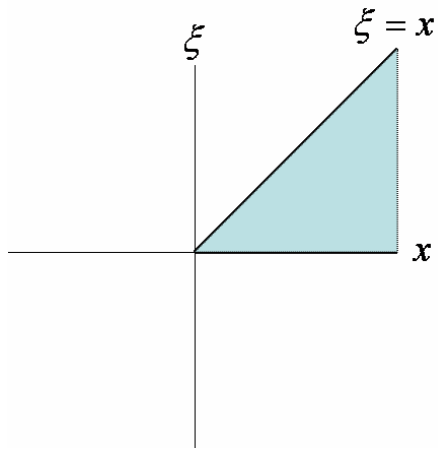
$$= \frac{q^2\gamma^6}{6\pi\varepsilon_0 c}\left[\left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2 + \left(1-\beta^2\right)\dot{\vec{\beta}}\cdot\dot{\vec{\beta}}\right]$$

$$\left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2 = \beta^2\dot{\vec{\beta}}\cdot\dot{\vec{\beta}} - \left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2$$

$$= \frac{q^2\gamma^6}{6\pi\varepsilon_0 c}\left[\left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2 + \dot{\vec{\beta}}\cdot\dot{\vec{\beta}} - \left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2 - \left(\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2\right]$$

$$= \frac{q^2\gamma^6}{6\pi\varepsilon_0 c}\left[\dot{\vec{\beta}}\cdot\dot{\vec{\beta}} - \left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2\right]$$

5. The key to this problem is to realize that when you interchange the order of integration, one must re-express the new limits of integration properly. Clearly, the range of integration in the $\xi - x$ plane is the blue wedge extending to positive infinity,



so

$$P = \frac{\sqrt{3}}{8\pi^2\varepsilon_0}\frac{e^2}{\rho}\omega_c\gamma\int_0^\infty \xi\int_\xi^\infty K_{5/3}(x)\,dx\,d\xi = \frac{\sqrt{3}}{8\pi^2\varepsilon_0}\frac{e^2}{\rho}\omega_c\gamma\int_0^\infty K_{5/3}(x)\int_0^x \xi\,d\xi\,dx$$

$$= \frac{\sqrt{3}}{16\pi^2\varepsilon_0}\frac{e^2}{\rho}\omega_c\gamma\int_0^\infty x^2 K_{5/3}(x)\,dx = \frac{e^2 c}{6\pi\varepsilon_0\rho^2}\gamma^4$$

A similar calculation allows evaluation of the average energy of the emitted photons

$$\langle E_\gamma\rangle = \frac{\int \hbar\omega\frac{d\dot{n}_\gamma}{d\omega}d\omega}{\int\frac{d\dot{n}_\gamma}{d\omega}d\omega} = \frac{\hbar\omega_c\int_0^\infty \xi\int_\xi^\infty K_{5/3}(x)\,dx\,d\xi}{\int_0^\infty\int_\xi^\infty K_{5/3}(x)\,dx\,d\xi} = \frac{\frac{16\pi}{2\cdot 9\sqrt{3}}}{\frac{5\pi}{3}}\hbar\omega_c = \frac{8}{15\sqrt{3}}\hbar\omega_c\,.$$

6. Normalization of the Gaussian-elliptical phase space distribution means

$$\int\int \rho(x,x')\,dx\,dx' = A\int\int \exp\left(-\left(\gamma x^2 + 2\alpha x x' + \beta x'^2\right)/2\varepsilon\right)dx\,dx' = 1$$

$$\int\int \exp\left(-\left(\gamma x^2 + 2\alpha x x' + \beta x'^2\right)/2\varepsilon\right)dx\,dx'$$

$$= \int\int \exp\left[-\frac{\gamma}{2\varepsilon}\left(x + \frac{\alpha}{\gamma}x'\right)^2 + \frac{\alpha^2}{2\gamma\varepsilon}x'^2 - \frac{\beta}{2\varepsilon}x'^2\right]dx\,dx'$$

$$= \sqrt{2\pi}\sqrt{\frac{\varepsilon}{\gamma}}\int \exp\left[\frac{\alpha^2}{2\gamma\varepsilon}x'^2 - \frac{\beta}{2\varepsilon}x'^2\right]dx' = 2\pi\sqrt{\frac{\varepsilon}{\gamma}}\sqrt{\varepsilon\gamma} = 2\pi\varepsilon$$

$$\therefore A = \frac{1}{2\pi\varepsilon}$$

Easiest to evaluate $\langle x'^2\rangle$ first

$$\left\langle x'^2 \right\rangle = \int\int x'^2 \rho\left(x, x'\right) dx dx' =$$

$$= \frac{1}{2\pi\varepsilon} \int\int x'^2 \exp\left[-\frac{\gamma}{2\varepsilon}\left(x + \frac{\alpha}{\gamma} x' \right)^2 + \frac{\alpha^2}{2\gamma\varepsilon} x'^2 - \frac{\beta}{2\varepsilon} x'^2 \right] dx dx'$$

$$= \frac{\sqrt{2\pi}}{2\pi\varepsilon} \sqrt{\frac{\varepsilon}{\gamma}} \int x'^2 \exp\left[\frac{\alpha^2}{2\gamma\varepsilon} x'^2 - \frac{\beta}{2\varepsilon} x'^2 \right] dx' = \frac{2\pi}{2\pi\varepsilon} \sqrt{\frac{\varepsilon}{\gamma}} \left(\varepsilon\gamma\right)^{3/2} = \varepsilon\gamma$$

Use this to evaluate $\left\langle x^2 \right\rangle$

$$\left\langle x^2 \right\rangle = \int\int x^2 \rho\left(x, x'\right) dx dx'$$

$$= \frac{1}{2\pi\varepsilon} \int\int x^2 \exp\left[-\frac{\gamma}{2\varepsilon}\left(x + \frac{\alpha}{\gamma} x' \right)^2 + \frac{\alpha^2}{2\gamma\varepsilon} x'^2 - \frac{\beta}{2\varepsilon} x'^2 \right] dx dx'$$

$$= \frac{1}{2\pi\varepsilon} \int\int \left(x - \frac{\alpha}{\gamma} x' \right)^2 \exp\left[-\frac{\gamma}{2\varepsilon} x^2 + \frac{\alpha^2}{2\gamma\varepsilon} x'^2 - \frac{\beta}{2\varepsilon} x'^2 \right] dx dx'$$

$$= \frac{1}{2\pi\varepsilon} \int\int \left(x^2 - \frac{2\alpha}{\gamma} x x' + \frac{\alpha^2}{\gamma^2} x'^2 \right) \exp\left[-\frac{\gamma}{2\varepsilon} x^2 + \frac{\alpha^2}{2\gamma\varepsilon} x'^2 - \frac{\beta}{2\varepsilon} x'^2 \right] dx dx'$$

$$= \frac{\sqrt{2\pi}}{2\pi\varepsilon} \frac{\varepsilon}{\gamma} \sqrt{\frac{\varepsilon}{\gamma}} \sqrt{2\pi} \sqrt{\varepsilon\gamma} + 0 + \frac{\sqrt{2\pi}}{2\pi\varepsilon} \sqrt{\frac{\varepsilon}{\gamma}} \frac{\alpha^2}{\gamma^2} \sqrt{2\pi} \left(\varepsilon\gamma\right)^{3/2} = \frac{\varepsilon}{\gamma}\left(1 + \alpha^2\right) = \varepsilon\beta$$

and the cross term $\left\langle x x' \right\rangle$

$$\left\langle x x' \right\rangle = \int\int x x' \rho\left(x, x'\right) dx dx' = \frac{1}{2\pi\varepsilon} \int\int x x' \exp\left[-\frac{\gamma}{2\varepsilon}\left(x + \frac{\alpha}{\gamma} x' \right)^2 + \frac{\alpha^2}{2\gamma\varepsilon} x'^2 - \frac{\beta}{2\varepsilon} x'^2 \right] dx dx'$$

$$= \frac{1}{2\pi\varepsilon} \int\int \left(x - \frac{\alpha}{\gamma} x' \right) x' \exp\left[-\frac{\gamma}{2\varepsilon} x^2 + \frac{\alpha^2}{2\gamma\varepsilon} x'^2 - \frac{\beta}{2\varepsilon} x'^2 \right] dx dx' = 0 + \left(-\frac{\alpha}{\gamma} \right) \gamma\varepsilon = -\alpha\varepsilon$$

Clearly $\varepsilon_{rms} = \sqrt{\varepsilon\beta\varepsilon\gamma - \alpha^2\varepsilon^2} = \varepsilon.$