Physics 417/517
Homework 3 Solutions

1. By taking the trace of the longitudinal 1-turn matrix in the discussion of the microtron phase stability we have
$$\cos\mu = 1 - \frac{2\pi^2 \rho_l e V_c}{\lambda E_l}\sin\phi_s = 1 - \frac{2\pi^2 \rho_l e V_c \cos\phi_s}{\lambda E_l}\tan\phi_s$$
As in the slide on the microtron phase stability condition, for velocity of light electrons $\rho = c/(2\pi f_c)$ and $f_{RF} = c/\lambda$. Substituting yields
$$\cos\mu = 1 - \frac{\pi f_{RF} e V_c \cos\phi_s}{f_c m c^2}\tan\phi_s = 1 - \pi\nu\tan\phi_s,$$
because $\nu = f_{RF}\Delta\gamma / f_c$ in the microtron. A straightforward calculation yields $\mu = 63.5^\circ$, or 1.11 rad, for the phase advance. Therefore, it takes a little under six turns for a full longitudinal oscillation.

2. This is several pages of algebra and somewhat tedious. However, as a result of this calculation you should be convinced that the formula for the transfer matrix in terms of $\alpha, \beta,$ and $\Delta\mu$ is accurate. If it weren't accurate, there's no way that all the matrix elements would conspire nicely to satisfy the composition formula!

$$M_{s'',s} = M_{s'',s'} M_{s',s} =$$
$$\begin{pmatrix} \sqrt{\frac{\beta(s'')}{\beta(s')}}\left(\cos\Delta\mu_{s'',s'} + \alpha(s')\sin\Delta\mu_{s'',s'}\right) & \sqrt{\beta(s'')\beta(s')}\sin\Delta\mu_{s'',s'} \\ -\frac{1}{\sqrt{\beta(s'')\beta(s')}}\begin{bmatrix}\left(1+\alpha(s'')\alpha(s')\right)\sin\Delta\mu_{s'',s'} \\ +\left(\alpha(s'')-\alpha(s')\right)\cos\Delta\mu_{s'',s'}\end{bmatrix} & \sqrt{\frac{\beta(s')}{\beta(s'')}}\left(\cos\Delta\mu_{s'',s'} - \alpha(s'')\sin\Delta\mu_{s'',s'}\right) \end{pmatrix}$$
$$\times\begin{pmatrix} \sqrt{\frac{\beta(s')}{\beta(s)}}\left(\cos\Delta\mu_{s',s} + \alpha(s)\sin\Delta\mu_{s',s}\right) & \sqrt{\beta(s')\beta(s)}\sin\Delta\mu_{s',s} \\ -\frac{1}{\sqrt{\beta(s')\beta(s)}}\begin{bmatrix}\left(1+\alpha(s')\alpha(s)\right)\sin\Delta\mu_{s',s} \\ +\left(\alpha(s')-\alpha(s)\right)\cos\Delta\mu_{s',s}\end{bmatrix} & \sqrt{\frac{\beta(s)}{\beta(s')}}\left(\cos\Delta\mu_{s',s} - \alpha(s')\sin\Delta\mu_{s',s}\right) \end{pmatrix}.$$

$$
\begin{pmatrix}
\sqrt{\dfrac{\beta(s'')}{\beta(s)}}\begin{bmatrix}\begin{array}{l}
\cos\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s}\\
+\alpha(s')\sin\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s}\\
+\alpha(s)\cos\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s}\\
+\alpha(s')\alpha(s)\sin\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s}\\
-\left[1+\alpha(s')\alpha(s)\right]\sin\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s}\\
-\left[\alpha(s')-\alpha(s)\right]\sin\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s}
\end{array}\end{bmatrix}
&
\sqrt{\beta(s'')\beta(s)}\begin{bmatrix}\begin{array}{l}
\cos\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s}\\
+\alpha(s')\sin\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s}\\
+\sin\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s}\\
-\alpha(s')\sin\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s}
\end{array}\end{bmatrix}
\\
-\dfrac{1}{\sqrt{\beta(s'')\beta(s)}}\begin{bmatrix}\begin{array}{l}
\left[1+\alpha(s'')\alpha(s')\right]\sin\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s}\\
+\left[\alpha(s'')-\alpha(s')\right]\cos\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s}\\
+\alpha(s)\left[1+\alpha(s'')\alpha(s')\right]\sin\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s}\\
+\alpha(s)\left[\alpha(s'')-\alpha(s')\right]\cos\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s}\\
+\left[1+\alpha(s')\alpha(s)\right]\cos\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s}\\
+\left[\alpha(s')-\alpha(s)\right]\cos\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s}\\
-\alpha(s'')\left[1+\alpha(s')\alpha(s)\right]\sin\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s}\\
-\alpha(s'')\left[\alpha(s')-\alpha(s)\right]\sin\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s}
\end{array}\end{bmatrix}
&
\sqrt{\dfrac{\beta(s)}{\beta(s'')}}\begin{bmatrix}\begin{array}{l}
-\left[1+\alpha(s'')\alpha(s')\right]\sin\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s}\\
-\left[\alpha(s'')-\alpha(s')\right]\cos\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s}\\
\cos\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s}\\
-\alpha(s'')\sin\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s}\\
-\alpha(s')\cos\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s}\\
+\alpha(s'')\alpha(s')\sin\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s}
\end{array}\end{bmatrix}
\end{pmatrix}=
$$

$$
\begin{pmatrix}
\sqrt{\dfrac{\beta(s'')}{\beta(s)}}\begin{bmatrix}\begin{array}{l}
\cos\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s}\\
+\alpha(s)\cos\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s}\\
-\sin\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s}\\
+\alpha(s)\sin\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s}
\end{array}\end{bmatrix}
&
\sqrt{\beta(s'')\beta(s)}\begin{bmatrix}\begin{array}{l}
\cos\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s}\\
+\sin\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s}
\end{array}\end{bmatrix}
\\
-\dfrac{1}{\sqrt{\beta(s'')\beta(s)}}\begin{bmatrix}\begin{array}{l}
\sin\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s}\\
+\alpha(s'')\cos\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s}\\
+\alpha(s)\sin\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s}\\
+\alpha(s)\alpha(s'')\cos\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s}\\
+\cos\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s}\\
-\alpha(s)\cos\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s}\\
-\alpha(s'')\sin\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s}\\
-\alpha(s'')\alpha(s)\sin\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s}
\end{array}\end{bmatrix}
&
\sqrt{\dfrac{\beta(s)}{\beta(s'')}}\begin{bmatrix}\begin{array}{l}
-\sin\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s}\\
-\alpha(s'')\cos\Delta\mu_{s'',s'}\sin\Delta\mu_{s',s}\\
\cos\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s}\\
-\alpha(s'')\sin\Delta\mu_{s'',s'}\cos\Delta\mu_{s',s}
\end{array}\end{bmatrix}
\end{pmatrix}=
$$

$$
\begin{pmatrix}
\sqrt{\frac{\beta(s")}{\beta(s)}}\begin{bmatrix}\cos\left(\Delta\mu_{s",s'}+\Delta\mu_{s',s}\right)\\ +\alpha(s)\sin\left(\Delta\mu_{s",s'}+\Delta\mu_{s',s}\right)\end{bmatrix} & \sqrt{\beta(s")\beta(s)}\sin\left(\Delta\mu_{s",s'}+\Delta\mu_{s',s}\right) \\
-\frac{1}{\sqrt{\beta(s")\beta(s)}}\left\{\begin{matrix}\left[1+\alpha(s")\alpha(s)\right]\sin\left(\Delta\mu_{s",s'}+\Delta\mu_{s',s}\right)\\ +\left[\alpha(s")-\alpha(s)\right]\cos\left(\Delta\mu_{s",s'}+\Delta\mu_{s',s}\right)\end{matrix}\right\} & \sqrt{\frac{\beta(s)}{\beta(s")}}\begin{bmatrix}\cos\left(\Delta\mu_{s",s'}+\Delta\mu_{s',s}\right)\\ -\alpha(s")\sin\left(\Delta\mu_{s",s'}+\Delta\mu_{s',s}\right)\end{bmatrix}
\end{pmatrix} =
$$

$$
\begin{pmatrix}
\sqrt{\frac{\beta(s")}{\beta(s)}}\left[\cos\Delta\mu_{s",s}+\alpha(s)\sin\Delta\mu_{s",s}\right] & \sqrt{\beta(s")\beta(s)}\sin\Delta\mu_{s",s} \\
-\frac{1}{\sqrt{\beta(s")\beta(s)}}\left\{\begin{matrix}\left[1+\alpha(s")\alpha(s)\right]\sin\Delta\mu_{s",s}\\ +\left[\alpha(s")-\alpha(s)\right]\cos\Delta\mu_{s",s}\end{matrix}\right\} & \sqrt{\frac{\beta(s)}{\beta(s")}}\left[\cos\Delta\mu_{s",s}-\alpha(s")\sin\Delta\mu_{s",s}\right]
\end{pmatrix}
$$

Now

$$
\begin{aligned}
\left(M_{s',s}\right)_{11} &= \sqrt{\frac{\beta(s')}{\beta(s)}}\left[\cos\Delta\mu_{s',s}+\alpha(s)\sin\Delta\mu_{s',s}\right] \\
\left(M_{s',s}\right)_{12} &= \sqrt{\beta(s')\beta(s)}\sin\Delta\mu_{s',s} \\
\beta(s)\left(M_{s',s}\right)_{11}-\alpha(s)\left(M_{s',s}\right)_{12} &= \sqrt{\beta(s')\beta(s)}\cos\Delta\mu_{s',s} \\
\therefore \tan\Delta\mu_{s',s} &= \frac{\left(M_{s',s}\right)_{12}}{\beta(s)\left(M_{s',s}\right)_{11}-\alpha(s)\left(M_{s',s}\right)_{12}}
\end{aligned}
$$

3. This time we'll build up the solutions from exponentials instead of sinh and cosh, as was done before. $D_{p,0}(s)$ satisfies

$$
\frac{d^2D_{p,0}}{ds^2}-(-k)D_{p,0}=\frac{1}{\rho}
$$

with boundary conditions $D_{p.0}(s=0)=0$ and $D'_{p.0}(s=0)=0$. First part of the solution is

$$
D_{p,0}(s)=\frac{1}{k\rho}+A_0\exp\left(\sqrt{-k}s\right)+B_0\exp\left(-\sqrt{-k}s\right)
$$

$$
D'_{p,0}(0)=0\rightarrow B_0=A_0;D_{p,0}(s)=0\rightarrow A_0=-\frac{1}{2k\rho}
$$

$$
\therefore D_{p,0}(s)=\frac{1}{k\rho}\left(1-\cosh\left(\sqrt{-k}s\right)\right)=\frac{1}{(-k)\rho}\left(\cosh\left(\sqrt{-k}s\right)-1\right)
$$

$$
D_{p,0}(s)=-\frac{\sqrt{-k}}{k\rho}\sinh\left(\sqrt{-k}s\right)=\frac{1}{\sqrt{-k}\rho}\sinh\left(\sqrt{-k}s\right)
$$

The general solution for the dispersion is

$$
D(s)=A\exp\left(\sqrt{-k}s\right)+B\exp\left(-\sqrt{-k}s\right)+D_{p,0}(s-s_0)
$$

$$
D'(s)=A\sqrt{-k}\exp\left(\sqrt{-k}s\right)-B\sqrt{-k}\exp\left(-\sqrt{-k}s\right)+D'_{p,0}(s-s_0)
$$

$$D(s_0) = A\exp\left(\sqrt{-k}s_0\right) + B\exp\left(-\sqrt{-k}s_0\right)$$

$$D'(s_0) = A\sqrt{-k}\exp\left(\sqrt{-k}s_0\right) - B\sqrt{-k}\exp\left(-\sqrt{-k}s_0\right)$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{-1}{\sqrt{-k}}\begin{pmatrix} -\sqrt{-k}\exp\left(-\sqrt{-k}s_0\right) & -\exp\left(-\sqrt{-k}s_0\right) \\ -\sqrt{-k}\exp\left(\sqrt{-k}s_0\right) & \exp\left(\sqrt{-k}s_0\right) \end{pmatrix}\begin{pmatrix} D(s_0) \\ D'(s_0) \end{pmatrix}$$

$$\therefore D(s) = \cosh\left(\sqrt{-k}(s-s_0)\right)D(s_0) + \sinh\left(\sqrt{-k}(s-s_0)\right)D'(s_0)/\sqrt{-k} + D_{p,0}(s-s_0)$$

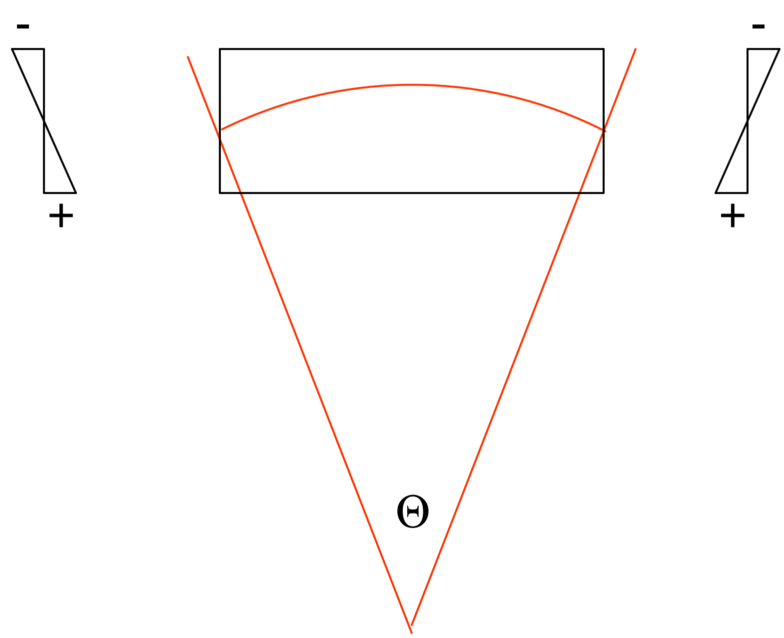
$$D'(s) = \sqrt{-k}\sinh\left(\sqrt{-k}(s-s_0)\right)D(s_0) + \cosh\left(\sqrt{-k}(s-s_0)\right)D'(s_0) + D'_{p,0}(s-s_0)$$

Putting $s - s_0 = L$ in these two equations gives the transfer matrix in the (corrected!) statement of the problem.

4. The 3×3 transfer matrix for a horizontal bend with bend angle Θ is

$$M_{\text{sector}} = \begin{pmatrix} \cos\Theta & \rho\sin\Theta & \rho(1-\cos\Theta) \\ -\sin\Theta/\rho & \cos\Theta & \sin\Theta \\ 0 & 0 & 1 \end{pmatrix}$$

The figure shows the geometry, + means magnetic field must be added to the sector magnet to get the rectangular magnet and – means magnetic field must be subtracted.



The transfer matrices for the wedges must have (12), (13), and (23) elements zero because, for small Θ the wedges are “thin”. The kick angle, by the wedge has

$$\left.\frac{dx}{ds}\right|_{after} = \left.\frac{dx}{ds}\right|_{before} + \frac{dL}{\rho} \approx \left.\frac{dx}{ds}\right|_{before} + \frac{x}{\rho}\tan\frac{\Theta}{2}$$

where dL is the extra path length through the wedge. The sign follows from the fact that positive x leads to less bending than the design orbit, meaning a positive dx/ds is generated. The wedge matrix is

$$M_{\text{wedge}} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\rho}\tan\frac{\Theta}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Multiplying out yields

$$M_{\text{tot}} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\rho}\tan\frac{\Theta}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\Theta & \rho\sin\Theta & \rho\left(1-\cos\Theta\right) \\ -\sin\Theta/\rho & \cos\Theta & \sin\Theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\rho}\tan\frac{\Theta}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\rho}\tan\frac{\Theta}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\Theta + \sin\Theta\tan\Theta/2 & \rho\sin\Theta & \rho\left(1-\cos\Theta\right) \\ \frac{1}{\rho}\left(-\sin\Theta + \cos\Theta\tan\frac{\Theta}{2}\right) & \cos\Theta & \sin\Theta \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \rho\sin\Theta & \rho\left(1-\cos\Theta\right) \\ 0 & 1 & 2\tan\frac{\Theta}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

This solution is somewhat different than Wille's final result as it is exact. It does reduce to his solution in the small angle approximation though. It agrees with Wiedemann's book *Particle Accelerator Physics I*.