Physics 417/517
Homework 2 Solutions

1. The frequency in time of the vertical oscillation is $\sqrt{n}\Omega_c / 2\pi$ and the revolution frequency is $\Omega_c / 2\pi$. Therefore, the time it takes to make 10 revolutions is $10\dfrac{2\pi}{\Omega_c}$. The number of vertical oscillations is the product of these two numbers $\sqrt{n}10 = 7.75$. The number of radial oscillations is obtained by replacing $\sqrt{n}$ by $\sqrt{1-n}$. The resulting number of oscillations is $\sqrt{0.4}10 = 6.32$. Several students solved the problem using our "path length" variable *s*. The oscillation angular "frequencies" in *s* are $\sqrt{n}/\rho$ and $\sqrt{1-n}/\rho$, and the "frequencies" in *s* are $\sqrt{n}/(2\pi\rho)$ and $\sqrt{1-n}/(2\pi\rho)$. The path length of 10 oscillations is $10(2\pi\rho)$. Of course, the final result does not depend on the variables in which it is evaluated.

2. This problem is a perfect example of the use of the magnetic rigidity. The relativistic momentum of the electrons is $p = \sqrt{\gamma^2 - 1}m_0 c = \sqrt{\gamma^2 - 1}(0.511\ \text{MeV}/c)$, the magnetic rigidity is $\sqrt{\gamma^2 - 1}(0.511\ \text{MV}/c)$, and 1 T = 1 (V sec)/(m^2). Now

$$B = \frac{(B\rho)}{L} 2\sin(\theta/2)$$

for magnets in the normal configuration. The bend angles are $\pi/16 = 0.19635$ rad for the first arc and $\pi/32 = 0.098175$ rad for the rest of the arcs.

$$B_1 = \frac{\sqrt{(605/0.511)^2 - 1} \times 0.511 \times 10^6\ \text{V}}{2.998 \times 10^8\ \text{m/sec}(1\ \text{m})} 2\sin 0.098175 = 0.3956\ \text{T} = 3.956\ \text{kG}$$

$$B_2 = \frac{\sqrt{(1693/0.511)^2 - 1} \times 0.511 \times 10^6\ \text{V}}{2.998 \times 10^8\ \text{m/sec}(1\ \text{m})} 2\sin 0.04909 = 0.5542\ \text{T} = 5.542\ \text{kG}$$

$$B_3 = \frac{\sqrt{(2781/0.511)^2 - 1} \times 0.511 \times 10^6\ \text{V}}{2.998 \times 10^8\ \text{m/sec}(2\ \text{m})} 2\sin 0.04909 = 0.4552\ \text{T} = 4.552\ \text{kG}$$

$$B_4 = \frac{\sqrt{(3868/0.511)^2 - 1} \times 0.511 \times 10^6\ \text{V}}{2.998 \times 10^8\ \text{m/sec}(3\ \text{m})} 2\sin 0.04909 = 0.4220\ \text{T} = 4.220\ \text{kG}$$

$$B_5 = \frac{\sqrt{(4956/0.511)^2 - 1} \times 0.511 \times 10^6\ \text{V}}{2.998 \times 10^8\ \text{m/sec}(3\ \text{m})} 2\sin 0.04909 = 0.5408\ \text{T} = 5.408\ \text{kG}$$

Arc	Electron Energy (MeV)	Number of Dipoles	Dipole Length (m)	Bend Angle (rad)	Magnetic Field (T)
1	605	16	1	0.19635	0.3956
2	1693	32	1	0.098175	0.5542
3	2781	32	2	0.098175	0.4552
4	3868	32	3	0.098175	0.4220
5	4956	32	3	0.098175	0.5408

3. Note the equation for the ellipse is

$$\frac{A}{C}x^2 + \frac{2B}{C}xy + y^2 = \frac{D}{C},$$

and one may compute the area of the region inside the ellipse as

$$Area = \iint dxdy$$

where to compute the integral, we need only specify properly the equation of the boundary of the elliptical region. If the *y*-extent of the ellipse is computed as a function of *x*, then

$$Area = \int_{x_{\min}}^{x_{\max}} \int_{y_-}^{y_+} dydx = \int_{x_{\min}}^{x_{\max}} \left(y_+ - y_-\right)dx$$

where the y_+ and y_- are the equations of the upper and lower boundaries of the elliptical region

$$y_\pm = \frac{-2Bx/C \pm \sqrt{\left(2Bx/C\right)^2 - 4\left(Ax^2/C - D/C\right)}}{2}$$
$$= -Bx/C \pm \sqrt{\left(Bx/C\right)^2 - \left(Ax^2/C - D/C\right)}$$

The limits in *x* are clearly where the two curves meet, i.e., those *x*-values where the discriminate of the quadratic solution vanishes:

$$\frac{\left(AC - B^2\right)}{C^2}x^2 = \frac{D}{C} \rightarrow x_{\max} = -x_{\min} = \sqrt{\frac{DC}{AC - B^2}}$$

Consequently

$$Area = \int_{x_{\min}}^{x_{\max}} 2\sqrt{\frac{D}{C} - \frac{AC - B^2}{C^2}x^2}\,dx$$
$$= 2\sqrt{\frac{D}{C}} \int_{-x_{\max}}^{x_{\max}} \sqrt{1 - \frac{x^2}{x_{\max}^2}}\,dx$$
$$= 2\sqrt{\frac{D}{C}}x_{\max} \int_{-1}^{1} \sqrt{1 - X^2}\,dX$$

The final integral is evaluated by trigonometric substitution $X = \cos\theta$. The value is $\pi/2$ (one-half the area of a circle of radius 1).

$$Area = 2\sqrt{\frac{D}{C}}x_{\max}\frac{\pi}{2} = \sqrt{\frac{D}{C}}\sqrt{\frac{DC}{AC-B^2}}\pi = \frac{\pi D}{\sqrt{AC-B^2}}.$$

One could have just as well evaluated the integral using the "area between two curves" method from elementary calculus. Also, one could have solved the quadratic for $x_{\pm}$ in as a function of *y* and sliced up the area horizontally, performing the *x* integral first. Clearly the procedure to evaluate the integrals would be identical, except the roles of *A* and *C* would be swapped. The final answer is identical under the interchange of *A* and *C*, as it should be!

4. Either $\cosh\sqrt{-k}s$ and $\sinh\sqrt{-k}s$ or $\exp\sqrt{-k}s$ and $\exp-\sqrt{-k}s$ could be used as the fundamental solutions of the homogeneous equation to build up the transfer matrix. Several students used each pair. In this particular solution let's use the pair $\cosh\sqrt{-k}s$ and $\sinh\sqrt{-k}s$.

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cosh\left(\sqrt{-k}s\right) & \sinh\left(\sqrt{-k}s\right) \\ \sqrt{-k}\sinh\left(\sqrt{-k}s\right) & \sqrt{-k}\cosh\left(\sqrt{-k}s\right) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$$

$$\begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix} = \begin{pmatrix} \cosh\left(\sqrt{-k}s_0\right) & \sinh\left(\sqrt{-k}s_0\right) \\ \sqrt{-k}\sinh\left(\sqrt{-k}s_0\right) & \sqrt{-k}\cosh\left(\sqrt{-k}s_0\right) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{\sqrt{-k}}\begin{pmatrix} \sqrt{-k}\cosh\left(\sqrt{-k}s_0\right) & -\sinh\left(\sqrt{-k}s_0\right) \\ -\sqrt{-k}\sinh\left(\sqrt{-k}s_0\right) & \cosh\left(\sqrt{-k}s_0\right) \end{pmatrix}\begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix}$$

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cosh\left(\sqrt{-k}s\right) & \sinh\left(\sqrt{-k}s\right) \\ \sqrt{-k}\sinh\left(\sqrt{-k}s\right) & \sqrt{-k}\cosh\left(\sqrt{-k}s\right) \end{pmatrix}$$

$$\times\frac{1}{\sqrt{-k}}\begin{pmatrix} \sqrt{-k}\cosh\left(\sqrt{-k}s_0\right) & -\sinh\left(\sqrt{-k}s_0\right) \\ -\sqrt{-k}\sinh\left(\sqrt{-k}s_0\right) & \cosh\left(\sqrt{-k}s_0\right) \end{pmatrix}\begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix}$$

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} \cosh\left(\sqrt{-k}(s-s_0)\right) & \sinh\left(\sqrt{-k}(s-s_0)\right)/\sqrt{-k} \\ \sqrt{-k}\sinh\left(\sqrt{-k}(s-s_0)\right) & \sqrt{-k}\cosh\left(\sqrt{-k}(s-s_0)\right) \end{pmatrix}\begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix}.$$

Using $s - s_0 = L$ for a defocusing quad of length *L*, the transfer matrix is

$$\begin{pmatrix} \cosh\left(\sqrt{-k}L\right) & \sinh\left(\sqrt{-k}L\right)/\sqrt{-k} \\ \sqrt{-k}\sinh\left(\sqrt{-k}L\right) & \sqrt{-k}\cosh\left(\sqrt{-k}L\right) \end{pmatrix}$$

5.

$$M=\begin{pmatrix}1 & 0\\ -1/(2f) & 1\end{pmatrix}\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1/f & 1\end{pmatrix}\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -1/(2f) & 1\end{pmatrix}$$

$$=\begin{pmatrix}1 & L\\ -1/(2f) & 1-L/(2f)\end{pmatrix}\begin{pmatrix}1 & 0\\ 1/f & 1\end{pmatrix}\begin{pmatrix}1-L/(2f) & L\\ -1/(2f) & 1\end{pmatrix}$$

$$=\begin{pmatrix}1+L/f & L\\ 1/(2f)-L/(2f^2) & 1-L/(2f)\end{pmatrix}\begin{pmatrix}1-L/(2f) & L\\ -1/(2f) & 1\end{pmatrix}$$

$$=\begin{pmatrix}1+L/(2f)-L^2/(2f^2)-L/(2f) & 2L+L^2/f\\ \begin{pmatrix}1/(2f)-L/(2f^2)-L/(4f^2)+L^2/(4f^3)\\ -1/(2f)+L/(4f^2)\end{pmatrix} & L/(2f)-L^2/(2f^2)+1-L/(2f)\end{pmatrix}$$

$$=\begin{pmatrix}1-L^2/(2f^2) & 2L+L^2/f\\ -L/(2f^2)+L^2/(4f^3) & 1-L^2/(2f^2)\end{pmatrix}.$$

To get the matrix starting with the middle of the defocusing magnet replace f with $-f$. The resulting transfer matrix is

$$M=\begin{pmatrix}1 & 0\\ 1/(2f) & 1\end{pmatrix}\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -1/f & 1\end{pmatrix}\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1/(2f) & 1\end{pmatrix}$$

$$=\begin{pmatrix}1-L^2/(2f^2) & 2L-L^2/f\\ -L/(2f^2)-L^2/(4f^3) & 1-L^2/(2f^2)\end{pmatrix}$$

Clearly the matrix traces of the two resulting matrices are the same.