Physics 417/517
Homework 1 Solutions

1. $m_p c^2 = 1.673\times10^{-27}\,\text{kg}\left(2.998\times10^{8}\ \text{m/sec}\right)^2 = 1.5037\times10^{-10}\ \text{J} = 938.5\ \text{MeV}$

$$m_d c^2 = 3.344\times10^{-27}\,\text{kg}\left(2.998\times10^{8}\ \text{m/sec}\right)^2 = 3.0037\times10^{-10}\ \text{J} = 1875.9\ \text{MeV}$$

$$\beta = \sqrt{1-1/\gamma^2}$$

$$\gamma = 2 \to \beta = \sqrt{3}/2 \to \text{v} = 2.596\times10^{8}\ \text{m/sec};\quad \gamma = 10 \to \beta = 0.995 \to \text{v} = 2.983\times10^{8}\ \text{m/sec}$$

$$\gamma = 2 \to KE = (2-1)m_0c^2,\ KE_p = 938.5\ \text{MeV},\ KE_d = 1875.9\ \text{MeV}$$

$$\gamma = 10 \to KE = (10-1)m_0c^2,\ KE_p = 8446.5\ \text{MeV},\ KE_d = 16833\ \text{MeV}$$

2. The key to this problem is to realize that the βs in the expression for γ have a time derivative in them. So an argument of this type is valid

$$\int_{t_1}^{t_2}\sqrt{1-\beta^2}\,dt = \int_{t_1}^{t_2}\sqrt{dt^2-\left(1/c^2\right)\left(dx^2+dy^2+dz^2\right)}.$$

But the invariance of the space-time interval shows that under a general Lorentz transformation

$$\int_{t_1}^{t_2}\sqrt{dt^2-\left(1/c^2\right)\left(dx^2+dy^2+dz^2\right)} = \int_{t'_1}^{t'_2}\sqrt{dt'^2-\left(1/c^2\right)\left(dx'^2+dy'^2+dz'^2\right)}$$

Therefore

$$\int_{t_1}^{t_2}\sqrt{1-\beta^2}\,dt = \int_{t'_1}^{t'_2}\sqrt{1-\beta'^2}\,dt'.$$

The square root differential expression is well defined because the four-velocity of any massive particle is always time-like.

A more formal and precise argument is the following. Divide the closed interval $[t_1, t_2]$ into N equal sub-intervals of duration $\Delta t = (t_2 - t_1)/N$ labeled by the index i: $I_i = [t_1 + (i-1)\Delta t, t_1 + i\Delta t]$. By the general Lorentz transformation between frames we may establish the coordinates of the space-time events $c(t_1), \vec{x}(t_1)$ and $c(t_1 + i\Delta t), \vec{x}(t_1 + i\Delta t)$ in the prime frame. Call the coordinates in the K' frame $c(t_0'), \vec{x}_0'$ and $ct_i', \vec{x}_i'$ respectively. Recall the mean value theorem from calculus

$$\int_{t_1}^{t_2} \sqrt{1-\beta^2}\, dt = \sum_{i=1}^{N} \int_{I_i} \sqrt{1-\beta^2}\, dt = \sum_{i=1}^{N} \sqrt{1-\beta^2(T_i)}\Delta t$$

for some $T_i \in I_i$. In the limit $N \to \infty$, the intervals become infinitesimals and

$$\vec{\beta}(T_i) \to \frac{\Delta \vec{x}_i}{\Delta t} \equiv \frac{\vec{x}_i - \vec{x}_{i-1}}{\Delta t}.$$

This means

$$\int_{t_1}^{t_2} \sqrt{1-\beta^2}\, dt = \lim_{N\to\infty} \sum_{i=1}^{N} \sqrt{\Delta t^2 - (1/c^2)|\Delta \vec{x}_i|^2}$$

The Lorentz transformations are linear, and so the differentials $c\Delta t$ and $\Delta \vec{x}$ transform in the same way as ct and $\vec{x}$. Therefore, by the invariance of the space-time interval under Lorentz transformations,

$$\int_{t_1}^{t_2} \sqrt{1-\beta^2}\, dt = \lim_{N\to\infty} \sum_{i=1}^{N} \sqrt{\Delta t_i'^2 - (1/c^2)|\Delta \vec{x}_i'|^2} = \int_{t_1'}^{t_2'} \sqrt{1-\beta'^2}\, dt',$$

because the durations of all the intervals in the prime frame approach zero as N increases without bound. It should be noted that $\Delta t_i'$ is NOT necessarily constant as i changes when there is acceleration in the orbit.

3.

$$\frac{d(\gamma m \mathrm{v}_x)}{d\tau} = qE_x u^0 + 0 + qcB_z u^2 - qcB_y u^3$$

$$\gamma \frac{d(\gamma m \mathrm{v}_x)}{dt} = qE_x \gamma + q\mathrm{v}_y B_z \gamma - q\mathrm{v}_z B_y \gamma$$

$$\frac{d(\gamma m \mathrm{v}_x)}{dt} = q\left(\vec{E} + \vec{\mathrm{v}} \times \vec{B}\right)_x$$

$$\frac{d\left(\gamma m \mathrm{v}_y\right)}{d\tau} = qE_y u^0 - qcB_z u^1 + 0 + qcB_x u^3$$

$$\gamma \frac{d\left(\gamma m \mathrm{v}_y\right)}{dt} = qE_y \gamma - q\mathrm{v}_x B_z \gamma + q\mathrm{v}_z B_x \gamma$$

$$\frac{d\left(\gamma m \mathrm{v}_y\right)}{dt} = q\left(\vec{E} + \vec{\mathrm{v}} \times \vec{B}\right)_y$$

$$\frac{d\left(\gamma m \mathrm{v}_z\right)}{d\tau} = qE_z u^0 + qcB_y u^1 - qcB_x u^2 + 0$$

$$\gamma \frac{d\left(\gamma m \mathrm{v}_z\right)}{dt} = qE_z \gamma + q\mathrm{v}_x B_y \gamma - q\mathrm{v}_y B_x \gamma$$

$$\frac{d\left(\gamma m \mathrm{v}_z\right)}{dt} = q\left(\vec{E} + \vec{\mathrm{v}} \times \vec{B}\right)_z$$

4. The non-relativistic cyclotron angular frequency is $\Omega_c = eB/m$

   For proton $= 1.602 \times 10^{-19}\,\mathrm{C}\ 0.3\ \mathrm{T}/1.673 \times 10^{-27}\ \mathrm{kg} = 28.73\ \mathrm{MHz}$

   For deuteron $= 1.602 \times 10^{-19}\,\mathrm{C}\ 0.3\ \mathrm{T}/3.344 \times 10^{-27}\ \mathrm{kg} = 14.37\ \mathrm{MHz}$

   The actual RF frequency for resonance is $\Omega_c / 2\pi$

   For proton = 4.572 MHz; For deuteron = 2.287 MHz

5. My solution

$$\frac{d\gamma}{dt} = 0 \rightarrow \frac{d\left|\vec{\mathrm{v}}\right|}{dt} = 0 \quad \left|\vec{\mathrm{v}}\right| = const = \mathrm{v}_0$$

$$\frac{d\mathrm{v}_x}{dt} = \frac{qB}{\gamma m}\mathrm{v}_y \qquad \frac{d\mathrm{v}_y}{dt} = -\frac{qB}{\gamma m}\mathrm{v}_x$$

$$\frac{d^2\mathrm{v}_x}{dt^2} + \Omega_c^2 \mathrm{v}_x = 0 \qquad \frac{d^2\mathrm{v}_y}{dt^2} + \Omega_c^2 \mathrm{v}_y = 0$$

$$\mathrm{v}_x(t) = A\cos\left(\Omega_c t + \delta\right) \qquad \mathrm{v}_y(t) = -A\sin\left(\Omega_c t + \delta\right)$$

$$\left|\vec{\mathrm{v}}\right| = A \rightarrow A = \mathrm{v}_0$$

$$x(t) = x_c + \frac{\mathrm{v}_0}{\Omega_c}\sin\left(\Omega_c t + \delta\right) \qquad y(t) = y_c + \frac{\mathrm{v}_0}{\Omega_c}\cos\left(\Omega_c t + \delta\right)$$

$$r = \frac{\mathrm{v}_0}{\Omega_c} = \frac{\beta c}{qB/\gamma m}$$

It turns out that Serkan Golge doesn't like me to invoke "unphysical" things like time derivatives of acceleration. For his benefit one can also get it like this

$$\mathrm{v}_x(t) = C + \Omega_c y(t)$$

$$\frac{d\mathrm{v}_y}{dt} = -\frac{qB}{\gamma m}\left(C + \Omega_c y(t)\right)$$

$$\frac{d^2 y}{dt^2} + \Omega_c^2 y = -\frac{qB}{\gamma m} C$$

$$y(t) = -\frac{C}{\Omega_c} + A' \cos\left(\Omega_c t + \delta\right)$$

$$\mathrm{v}_x(t) = \Omega_c A' \cos\left(\Omega_c t + \delta\right) \quad \mathrm{v}_y(t) = -\Omega_c A' \sin\left(\Omega_c t + \delta\right) \qquad \therefore \Omega_c A' = \mathrm{v}_0$$

$$x(t) = C' + A' \sin\left(\Omega_c t + \delta\right)$$

$$A' = r = \frac{\mathrm{v}_0}{\Omega_c}$$