Final Examination Solution
Physics 417/517

1. We have used the Advanced Photon Source (APS) at Argonne as an example in many of the homework problems. As was done in the midterm for LHC, let's investigate the parameters of the machine. The APS stores a nominal 100 mA beam current at the design energy of 7 GeV, or 7000 MeV.
   a. What is the relativistic γ of the electrons? What is their relativistic β?

$$\gamma = \frac{7000 \text{ MeV}}{0.511 \text{ MeV}} = 13698.63 \qquad \beta = \sqrt{1 - 1/\gamma^2} = 0.9999999973$$

   b. What is their magnetic rigidity?

$$B\rho = \frac{p}{e} = \frac{\beta\gamma\left(0.511 \text{ MV}\right)}{2.998\times10^8 \text{ m/sec}} = 23.349\frac{\text{V sec}}{\text{m}} = 23.349 \text{ T m}$$

   c. There are 81 dipole magnets of length 3.06 m, bending in normal configuration. What is the required dipole magnetic field?

$$B = \frac{2B\rho}{L}\sin\left(\theta/2\right) = \frac{2\cdot 23.349 \text{ T m}}{3.06 \text{ m}}\sin\left(\frac{\pi}{81}\right) = 0.592 \text{ T}$$

   d. What is the bend radius when the particles are in the dipoles?

$$\rho = \frac{23.349 \text{ T m}}{5.92 \text{ T}} = 39.46 \text{ m}$$

   e. If the machine circumference is 1104.00 m, what is the revolution frequency?

$$f = \frac{2.998\times10^8 \text{ m/sec}}{1104 \text{ m}} = 271.558 \text{ kHz}$$

   f. The RF frequency is reported to be 351.9 MHz. What is the harmonic number to four significant digits?

$$h = \frac{351.9 \text{ MHz}}{271.558 \text{ kHz}} = 1296$$

   g. What is the energy loss per electron per turn from radiation in the bend dipoles?

$$\delta E = \frac{4\pi r_e}{3\rho}\beta^3\gamma^3 E_{beam} = \frac{4\pi\cdot 2.82\times10^{-15}\text{ m}}{3\cdot 39.46\text{ m}}(13.698.63)^3\, 7\text{ GeV} = 5.387\text{ MeV}$$

h. The total energy gain per pass available from the RF system is 9.5 MV. What must the synchronous phase be in order to have energy equilibrium in the storage ring?

$$9.5\text{ MV}\cos\phi_s = 5.387\text{ MV} \rightarrow \cos\phi_s = \frac{5.387}{9.5} \rightarrow \phi_s = \pm 55.4^\circ$$

The sign is chosen to ensure longitudinal stability in part k.

i. The momentum compaction factor is 2.82 $\times 10^{-4}$. What is the transition energy?

$$\frac{1}{\gamma_{tr}^2} = 2.82\times10^{-4} \rightarrow \gamma_{tr} = 59.5 \qquad E_{tr} = 59.5\cdot 0.511\text{ MeV} = 30.4\text{ MeV}$$

j. What is the frequency slip factor η_c at 7 GeV?

$$\eta_c = \frac{1}{\gamma^2} - \frac{1}{\gamma_{tr}^2} \approx -2.82\times10^{-4}\text{ because } 13698.63 \gg 59.5$$

k. What is the small amplitude synchrotron oscillation frequency at the synchronous phase calculated in h?

$$f_s = \frac{\omega_s}{2\pi} = f_{rev}\sqrt{-\frac{h\eta_c}{2\pi}\frac{eV_c\sin\phi_s}{pc}} = 276.6\text{ kHz}\sqrt{\frac{1296\cdot 2.82\times10^{-4}}{2\pi}\frac{9.5\text{ MeV}\sin 55.4^\circ}{7\text{ GeV}}}$$

$$= 276.6\text{ kHz}\cdot 0.00806 = 2.19\text{ kHz} \qquad \text{"Longitudinal Tune"} = 0.008\text{ in APS Table}$$

l. What is the total power radiated by all the electrons in the bends?

$$P_{bends} = \frac{I_{beam}}{e}\delta E = 100\text{ mA}\cdot 5.387\text{ MV} = 0.5387\text{ MW}$$

Note: on part i, use the definition of momentum compaction in the 10/23 lecture. This slide was updated after some of you saw it the first time, to be consistent with Wille.

2. Summarize the properties of bend radiation. In this problem you may assume that the power frequency distribution of bend radiation

$$\frac{dP}{d\omega} = \frac{\sqrt{3}}{8\pi^2\varepsilon_0}\frac{e^2}{\rho}\gamma\frac{\omega}{\omega_c}\int_{\omega/\omega_c}^{\infty} K_{5/3}(x)\,dx$$

is known.

a. Derive the total power formula

$$P(t)=\frac{e^2c}{6\pi\varepsilon_0\rho^2}\beta^4\gamma^4$$

from the Lienard formula

$$P(t)=\frac{e^2}{6\pi\varepsilon_0 c}\gamma^6\left(\dot{\vec{\beta}}^2-\left[\vec{\beta}\times\dot{\vec{\beta}}\right]^2\right).$$

For uniform circular motion $\dot{\vec{\beta}}$ is perpendicular to $\vec{\beta}$ in the direction of the center of curvature. The magnitude is

$$\left|\dot{\vec{\beta}}\right|=\frac{\beta^2 c}{\rho}.$$

$$P(t)=\frac{e^2}{6\pi\varepsilon_0 c}\gamma^6\left[\frac{\beta^4c^2}{\rho^2}-\beta^2\frac{\beta^4c^2}{\rho^2}\right]=\frac{e^2c}{6\pi\varepsilon_0\rho^2}\beta^4\gamma^4$$

b. Show, by integrating the power frequency distribution over all frequencies, that the total power is consistent with the result from part a when $\beta\approx 1$.

$$P=\int_0^\infty\frac{dP}{d\omega}d\omega=\frac{\sqrt{3}}{8\pi^2\varepsilon_0}\frac{e^2}{\rho}\gamma\omega_c\int_0^\infty\xi\int_{\omega/\omega_c}^\infty K_{5/3}(x)\,dx\,d\xi$$

$$=\frac{\sqrt{3}}{8\pi^2\varepsilon_0}\frac{e^2}{\rho}\gamma\omega_c\int_0^\infty\int_0^x\xi K_{5/3}(x)\,d\xi\,dx=\frac{\sqrt{3}}{16\pi^2\varepsilon_0}\frac{e^2}{\rho}\gamma\omega_c\frac{16\pi}{9\sqrt{3}}$$

$$=\frac{e^2c}{6\pi\varepsilon_0\rho^2}\beta^4\gamma^4$$

c. Calculate the average energy of emitted photons photons in terms of the critical energy $\hbar\omega_c$.

$$\left\langle E_\gamma\right\rangle=\frac{\int\hbar\omega\frac{d\dot{n}_\gamma}{d\omega}d\omega}{\int\frac{d\dot{n}_\gamma}{d\omega}d\omega}=\frac{\hbar\omega_c\int_0^\infty\xi\int_\xi^\infty K_{5/3}(x)\,dx\,d\xi}{\int_0^\infty\int_\xi^\infty K_{5/3}(x)\,dx\,d\xi}=\frac{\frac{16\pi}{2\cdot 9\sqrt{3}}}{\frac{5\pi}{3}}\hbar\omega_c=\frac{8}{15\sqrt{3}}\hbar\omega_c$$

d. Calculate the *rms* spread of the emitted photons in terms of the critical energy $\hbar\omega_c$, where the spread is defined to be $\sqrt{\left\langle E^2\right\rangle-\left\langle E\right\rangle^2}$ and where E is the energy of the emitted photons.

$$\left\langle E_\gamma^2 \right\rangle = \frac{\int \hbar^2 \omega^2 \frac{d\dot{n}_\gamma}{d\omega} d\omega}{\int \frac{d\dot{n}_\gamma}{d\omega} d\omega} = \frac{\hbar^2 \omega_c^2 \int_0^\infty \xi^2 \int_\xi^\infty K_{5/3}(x)\, dx d\xi}{\int_0^\infty \int_\xi^\infty K_{5/3}(x)\, dx d\xi} = \frac{\frac{55\pi}{3\cdot 27}}{\frac{5\pi}{3}} \hbar\omega_c = \frac{11}{27} \hbar^2 \omega_c^2$$

$$E_{\gamma,rms} = \hbar\omega_c \sqrt{\frac{11}{27} - \frac{64}{25\cdot 27}} = \frac{\hbar\omega_c}{3\sqrt{3}} \sqrt{211}$$

3. The following matrix is unimodular

$$M = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}.$$

a. Find the two eigenvalues of the matrix. Call them λ_+ and λ_-.

$$\det(M - \lambda I) = \det \begin{pmatrix} 1/2 - \lambda & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 - \lambda \end{pmatrix} = \lambda^2 - \lambda + 1$$

$$\lambda_\pm = \frac{1 \pm \sqrt{1-4}}{2} = \cos 60^\circ \pm i \sin 60^\circ = e^{\pm i\pi/3}$$

b. Are the two eigenvalues real, or do they have some imaginary components?

Clearly there are some imaginary components.

c. Show $\lambda_+\lambda_- = 1$.

$$\lambda_+\lambda_- = e^{i\pi/3} e^{-i\pi/3} = 1$$

d. What is the phase advance of this matrix?

$$\mu = \cos^{-1} 1/2 = \pm 60^\circ, \quad \text{must choose } -60^\circ \text{ for positive } \beta$$

e. What are the $\alpha, \beta,$ and γ (Twiss parameters) for this matrix?

$$\alpha = \frac{1/2 - 1/2}{2\sin(-60^\circ)} = 0,\ \beta = \frac{-\sqrt{3}/2}{\sin(-60^\circ)} = \frac{-\sqrt{3}/2}{-\sqrt{3}/2} = 1,\ \gamma = \frac{-\sqrt{3}/2}{-\sqrt{3}/2} = 1$$

f. What is M^{100}, i. e., what are the individual matrix elements of the one hundredth power of M?

$$M^{100} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\left(-6000^\circ\right) + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\sin\left(-6000^\circ\right)$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\left(-240^\circ\right) + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\sin\left(-240^\circ\right)$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\left(120^\circ\right) + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\sin\left(120^\circ\right)$$
$$= \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$$

g. What is the one hundredth power of ξM where ξ is any real number, expressed in terms of the matrix found in f and powers of ξ?

$$\left(\xi M\right)^{100} = \xi^{100} M^{100} = \xi^{100} \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$$

h. What condition on $|\xi|$ guarantees that ξM, which is not necessarily unimodular, is stable, i.e., its matrix power remains bounded as the power increases?

M^n itself remains bounded as *n* increases.

$$\xi^n \to \infty \quad \text{if } |\xi| > 1$$
$$\xi^n \to 0 \quad \text{if } |\xi| < 1$$
$$\xi^n = \pm 1, \quad \text{that is a possible 180 phase shift only, if } |\xi| = 1$$

The condition for overall stability is $|\xi| \le 1$.

4. Recall our expressions from the homework for the transfer matrix of a FODO system. The one-period transfer matrix starting with the middle of the focusing magnet was

$$M_f = \begin{pmatrix} 1 & 0 \\ -1/(2f) & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/(2f) & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 - L^2/\left(2f^2\right) & 2L + L^2/f \\ -L/\left(2f^2\right) + L^2/\left(4f^3\right) & 1 - L^2/\left(2f^2\right) \end{pmatrix}.$$

and M_d is obtained from M_f by replacing f with $-f$

a. How should one choose *f*, in terms of *L*, so that the phase advance through one period of the FODO system is 45 degrees (corresponding to 1/8 of a transverse oscillation per period)?

$$\cos 45^\circ = \frac{1}{\sqrt{2}} = 1 - \frac{L^2}{2f^2}$$
$$2-\sqrt{2} = \frac{L^2}{f^2} \rightarrow f = \frac{L}{\sqrt{2-\sqrt{2}}}$$

b. For the matched phase space (x, x') ellipse in the 45 degree phase advance system, what is the beta-function in the middle of the focusing lens?

$$\sin 45^\circ = \frac{1}{\sqrt{2}}$$
$$\beta_f = \frac{2L + L^2/f}{\sin 45^\circ} = \sqrt{2}\left(2+\sqrt{2-\sqrt{2}}\right)L$$

c. For the matched phase space ellipse in the 45 degree phase advance system, what is the beta-function in the middle of the defocusing lens?

$$\beta_d = \frac{2L - L^2/f}{\sin 45^\circ} = \sqrt{2}\left(2-\sqrt{2-\sqrt{2}}\right)L$$

d. What are the alpha-functions for the matched ellipses at these same two locations?

Because $M_{11} = M_{22}$ in both cases, $\alpha_f = \alpha_d = 0$

e. Suppose L is 3 m, and the area of a matched phase space ellipse is $\pi\varepsilon = 2\pi \times 10^{-6}$ m radian, what are the maximum extents ($x_{\max}$) of the matched ellipses in the focusing and defocusing lenses of the 45 degrees phase advance system?

$$\beta_f = 3\cdot\sqrt{2}\cdot 2.765 \text{ m} = 11.73 \text{ m} \qquad \beta_d = 3\cdot\sqrt{2}\cdot 1.235 \text{ m} = 5.24 \text{ m}$$
$$x_{\max,f} = \sqrt{\varepsilon\beta_f} = 4.84 \text{ mm} \qquad x_{\max,d} = \sqrt{\varepsilon\beta_d} = 3.24 \text{ mm}$$

5. The Parabolic-elliptical phase space distribution is
$$\rho(x,x') = A\left(1-\left(\gamma x^2 + 2\alpha x x' + \beta x'^2\right)/\varepsilon\right)\Theta\left(1-\left(\gamma x^2 + 2\alpha x x' + \beta x'^2\right)/\varepsilon\right)$$
assuming $\beta\gamma - \alpha^2 = 1$.

I’ll give two solutions. The first is using “the standard” calculation. At the end I’ll show how the “Hint” solution is much easier and more expeditious.

a. Normalize the distribution.

$$1 = A\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}}\int_{-\frac{\alpha x}{\beta}-\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}}^{-\frac{\alpha x}{\beta}+\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}}\left(1-\left(\gamma x^2+2\alpha xx'+\beta x'^2\right)/\varepsilon\right)dx'dx =$$

$$A\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}}\frac{2\sqrt{\varepsilon\beta-x^2}}{\beta}\left(1-\frac{\gamma x^2}{\varepsilon}\right)dx - A\frac{2\alpha}{\varepsilon}\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}}x\frac{x'^2}{2}\Bigg|_{-\frac{\alpha x}{\beta}-\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}}^{-\frac{\alpha x}{\beta}+\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}}dx - A\frac{\beta}{\varepsilon}\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}}\frac{x'^3}{3}\Bigg|_{-\frac{\alpha x}{\beta}-\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}}^{-\frac{\alpha x}{\beta}+\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}}dx =$$

$$A\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}}\frac{2\sqrt{\varepsilon\beta-x^2}}{\beta}\left(1-\frac{\gamma x^2}{\varepsilon}\right)dx + A\frac{4\alpha}{\varepsilon}\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}}\frac{\alpha x^2}{\beta}\left(\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}\right)dx$$

$$-A\frac{\beta}{\varepsilon}\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}}2\frac{\alpha^2x^2}{\beta^2}\left(\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}\right)dx - A\frac{\beta}{\varepsilon}\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}}\frac{2}{3}\left(\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}\right)^3 dx =$$

$$A\int_{-\pi/2}^{\pi/2}2\sqrt{\frac{\varepsilon}{\beta}}\cos\theta\left(1-\beta\gamma\sin^2\theta\right)\sqrt{\varepsilon\beta}\cos\theta d\theta + A\frac{4\alpha^2}{\varepsilon\beta^2}(\varepsilon\beta)^{3/2}\int_{-\pi/2}^{\pi/2}\sin^2\theta\cos\theta\sqrt{\varepsilon\beta}\cos\theta d\theta$$

$$-A\frac{2\alpha^2}{\varepsilon\beta^2}(\varepsilon\beta)^{3/2}\int_{-\pi/2}^{\pi/2}\sin^2\theta\cos\theta\sqrt{\varepsilon\beta}\cos\theta d\theta - A\frac{2\beta}{3\varepsilon}\left(\sqrt{\frac{\varepsilon}{\beta}}\right)^3\int_{-\pi/2}^{\pi/2}(\cos\theta)^3\sqrt{\varepsilon\beta}\cos\theta d\theta =$$

$$A\left(\varepsilon\pi-\varepsilon\pi\beta\gamma\frac{1}{4}+\varepsilon\pi\alpha^2\frac{1}{2}-\varepsilon\pi\alpha^2\frac{1}{4}-\frac{2\varepsilon}{3}\frac{3}{8}\pi\right)=A\frac{\varepsilon\pi}{2} \qquad \therefore A=\frac{2}{\pi\varepsilon}$$

b. Calculate ε_{rms}.

$$\left\langle x^2\right\rangle = A\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}}\int_{-\frac{\alpha x}{\beta}-\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}}^{-\frac{\alpha x}{\beta}+\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}} x^2\left(1-\left(\gamma x^2+2\alpha xx'+\beta x'^2\right)/\varepsilon\right)dx'dx =$$

$$A\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}} x^2\,\frac{2\sqrt{\varepsilon\beta-x^2}}{\beta}\left(1-\frac{\gamma x^2}{\varepsilon}\right)dx - A\frac{2\alpha}{\varepsilon}\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}} x^3\,\frac{x'^2}{2}\bigg|_{-\frac{\alpha x}{\beta}-\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}}^{-\frac{\alpha x}{\beta}+\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}} dx - A\frac{\beta}{\varepsilon}\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}} x^2\,\frac{x'^3}{3}\bigg|_{-\frac{\alpha x}{\beta}-\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}}^{-\frac{\alpha x}{\beta}+\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}} dx =$$

$$A\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}} x^2\,\frac{2\sqrt{\varepsilon\beta-x^2}}{\beta}\left(1-\frac{\gamma x^2}{\varepsilon}\right)dx + A\frac{4\alpha}{\varepsilon}\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}}\frac{\alpha x^4}{\beta}\left(\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}\right)dx - A\frac{\beta}{\varepsilon}\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}} 2\frac{\alpha^2x^4}{\beta^2}\left(\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}\right)dx - A\frac{\beta}{\varepsilon}\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}}\frac{2x^2}{3}\left(\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}\right)^3 dx$$

$$= A\int_{-\pi/2}^{\pi/2} 2\sqrt{\frac{\varepsilon}{\beta}}\varepsilon\beta\sin^2\theta\cos\theta\left(1-\beta\gamma\sin^2\theta\right)\sqrt{\varepsilon\beta}\cos\theta d\theta + A\frac{4\alpha^2}{\varepsilon\beta^2}(\varepsilon\beta)^{5/2}\int_{-\pi/2}^{\pi/2}\sin^4\theta\cos\theta\sqrt{\varepsilon\beta}\cos\theta d\theta$$

$$-A\frac{2\alpha^2}{\varepsilon\beta^2}(\varepsilon\beta)^{5/2}\int_{-\pi/2}^{\pi/2}\sin^4\theta\cos\theta\sqrt{\varepsilon\beta}\cos\theta d\theta - A\frac{2\beta}{3\varepsilon}\left(\sqrt{\frac{\varepsilon}{\beta}}\right)^3\varepsilon\beta\int_{-\pi/2}^{\pi/2}\sin^2\theta\left(\cos\theta\right)^3\sqrt{\varepsilon\beta}\cos\theta d\theta =$$

$$A\varepsilon\beta\left(\varepsilon\pi\frac{1}{4}-\varepsilon\pi\beta\gamma\frac{1}{8}+\varepsilon\pi\alpha^2\frac{1}{4}-\varepsilon\pi\alpha^2\frac{1}{8}-\frac{2\varepsilon}{3}\frac{\pi}{16}\right) = A\frac{\varepsilon^2\beta\pi}{12} = \frac{\varepsilon\beta}{6}$$

$$\left\langle xx'\right\rangle = A\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}}\int_{-\frac{\alpha x}{\beta}-\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}}^{-\frac{\alpha x}{\beta}+\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}} xx'\left(1-\left(\gamma x^2+2\alpha xx'+\beta x'^2\right)/\varepsilon\right)dx'dx =$$

$$A\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}} x\frac{x'^2}{2}\bigg|_{-\frac{\alpha x}{\beta}-\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}}^{-\frac{\alpha x}{\beta}+\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}}\left(1-\frac{\gamma x^2}{\varepsilon}\right)dx - A\frac{2\alpha}{\varepsilon}\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}} x^2\frac{x'^3}{3}\bigg|_{-\frac{\alpha x}{\beta}-\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}}^{-\frac{\alpha x}{\beta}+\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}} dx - A\frac{\beta}{\varepsilon}\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}} x\frac{x'^4}{4}\bigg|_{-\frac{\alpha x}{\beta}-\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}}^{-\frac{\alpha x}{\beta}+\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}} dx =$$

$$-A\frac{\alpha}{\beta}\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}} x^2\,\frac{2\sqrt{\varepsilon\beta-x^2}}{\beta}\left(1-\frac{\gamma x^2}{\varepsilon}\right)dx - A\frac{2\alpha}{\varepsilon}\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}} 2\frac{\alpha^2x^4}{\beta^2}\left(\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}\right)dx - A\frac{2\alpha}{\varepsilon}\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}}\frac{2x^2}{3}\left(\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}\right)^3 dx$$

$$+A\frac{\beta}{\varepsilon}\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}} 2\frac{\alpha x^2}{\beta}\left(\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}\right)^3 dx + A\frac{\beta}{\varepsilon}\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}} 2\frac{\alpha^3x^4}{\beta^3}\left(\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}\right)dx = -A\frac{\alpha}{\beta}\int_{-\pi/2}^{\pi/2} 2\sqrt{\frac{\varepsilon}{\beta}}\varepsilon\beta\sin^2\theta\cos\theta\left(1-\beta\gamma\sin^2\theta\right)\sqrt{\varepsilon\beta}\cos\theta d\theta$$

$$-A\frac{4\alpha^3}{\varepsilon\beta^2}\sqrt{\frac{\varepsilon}{\beta}}(\varepsilon\beta)^2\int_{-\pi/2}^{\pi/2}\sin^4\theta\cos\theta\sqrt{\varepsilon\beta}\cos\theta d\theta - A\frac{4\alpha}{3\varepsilon}\left(\sqrt{\frac{\varepsilon}{\beta}}\right)^3\varepsilon\beta\int_{-\pi/2}^{\pi/2}\sin^2\theta\left(\cos\theta\right)^3\sqrt{\varepsilon\beta}\cos\theta d\theta$$

$$+A\frac{2\alpha}{\varepsilon}\varepsilon\beta\left(\sqrt{\frac{\varepsilon}{\beta}}\right)^3\int_{-\pi/2}^{\pi/2}\sin^2\theta\cos^3\theta\sqrt{\varepsilon\beta}\cos\theta d\theta + A\frac{2\alpha^3}{\varepsilon\beta^2}(\varepsilon\beta)^2\sqrt{\frac{\varepsilon}{\beta}}\int_{-\pi/2}^{\pi/2}\sin^4\theta\cos\theta\sqrt{\varepsilon\beta}\cos\theta d\theta =$$

$$A\varepsilon\alpha\left(-\varepsilon\pi\frac{1}{4}+\varepsilon\pi\beta\gamma\frac{1}{8}-\varepsilon\pi\alpha^2\frac{1}{4}-\frac{4}{3}\varepsilon\pi\frac{1}{16}+\varepsilon\pi\cdot\frac{1}{8}+\varepsilon\pi\alpha^2\frac{1}{8}\right) = -A\frac{\varepsilon^2\alpha\pi}{12} = -\frac{\varepsilon\alpha}{6}$$

$$\left\langle x'^2\right\rangle = A\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}}\int_{-\frac{\alpha x}{\beta}-\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}}^{-\frac{\alpha x}{\beta}+\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}} x'^2\left(1-\left(\gamma x^2+2\alpha xx'+\beta x'^2\right)/\varepsilon\right)dx'dx =$$

$$A\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}}\frac{x'^3}{3}\Bigg|_{-\frac{\alpha x}{\beta}-\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}}^{-\frac{\alpha x}{\beta}+\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}}\left(1-\frac{\gamma x^2}{\varepsilon}\right)dx - A\frac{2\alpha}{\varepsilon}\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}} x\frac{x'^4}{4}\Bigg|_{-\frac{\alpha x}{\beta}-\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}}^{-\frac{\alpha x}{\beta}+\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}} dx - A\frac{\beta}{\varepsilon}\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}}\frac{x'^5}{5}\Bigg|_{-\frac{\alpha x}{\beta}-\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}}^{-\frac{\alpha x}{\beta}+\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}} dx =$$

$$A\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}}\frac{2\alpha^2x^2}{\beta^2}\left(\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}\right)\left(1-\frac{\gamma x^2}{\varepsilon}\right)dx + A\frac{2}{3}\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}}\left(\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}\right)^3\left(1-\frac{\gamma x^2}{\varepsilon}\right)dx$$

$$+A\frac{2\alpha}{\varepsilon}\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}}\frac{2\alpha^3x^4}{\beta^3}\left(\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}\right)dx + A\frac{2\alpha}{\varepsilon}\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}}\frac{2\alpha x^2}{\beta}\left(\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}\right)^3 dx$$

$$-A\frac{\beta}{\varepsilon}\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}}2\frac{\alpha^4x^4}{\beta^4}\left(\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}\right)dx - A\frac{\beta}{\varepsilon}\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}}4\frac{\alpha^2x^2}{\beta^2}\left(\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}\right)^3 dx - A\frac{\beta}{\varepsilon}\int_{-\sqrt{\varepsilon\beta}}^{\sqrt{\varepsilon\beta}}\frac{2}{5}\left(\frac{\sqrt{\varepsilon\beta-x^2}}{\beta}\right)^5 dx =$$

$$A\frac{2\alpha^2}{\beta^2}\sqrt{\frac{\varepsilon}{\beta}}\varepsilon\beta\int_{-\pi/2}^{\pi/2}\sin^2\theta\cos\theta\left(1-\beta\gamma\sin^2\theta\right)\sqrt{\varepsilon\beta}\cos\theta d\theta + A\frac{2}{3}\left(\sqrt{\frac{\varepsilon}{\beta}}\right)^3\int_{-\pi/2}^{\pi/2}\cos^3\theta\left(1-\beta\gamma\sin^2\theta\right)\sqrt{\varepsilon\beta}\cos\theta d\theta$$

$$+A\frac{4\alpha^4}{\varepsilon\beta^3}(\varepsilon\beta)^2\sqrt{\frac{\varepsilon}{\beta}}\int_{-\pi/2}^{\pi/2}\sin^4\theta\cos\theta\sqrt{\varepsilon\beta}\cos\theta d\theta + A\frac{4\alpha^2}{\varepsilon\beta}\left(\sqrt{\frac{\varepsilon}{\beta}}\right)^3\varepsilon\beta\int_{-\pi/2}^{\pi/2}\sin^2\theta(\cos\theta)^3\sqrt{\varepsilon\beta}\cos\theta d\theta$$

$$-A\frac{2\alpha^4}{\varepsilon\beta^3}(\varepsilon\beta)^2\sqrt{\frac{\varepsilon}{\beta}}\int_{-\pi/2}^{\pi/2}\sin^4\theta\cos\theta\sqrt{\varepsilon\beta}\cos\theta d\theta - A\frac{4\alpha^2}{\varepsilon\beta}\varepsilon\beta\left(\sqrt{\frac{\varepsilon}{\beta}}\right)^3\int_{-\pi/2}^{\pi/2}\sin^2\theta\cos^3\theta\sqrt{\varepsilon\beta}\cos\theta d\theta$$

$$-A\frac{2\beta}{5\varepsilon}\left(\sqrt{\frac{\varepsilon}{\beta}}\right)^5\int_{-\pi/2}^{\pi/2}\cos^5\theta\sqrt{\varepsilon\beta}\cos\theta d\theta = A\varepsilon\left(\begin{array}{l}\frac{\varepsilon\pi}{\beta}\frac{\alpha^2}{4}-\frac{\varepsilon\pi}{\beta}\beta\gamma\frac{\alpha^2}{8}+\frac{2}{3}\frac{\varepsilon\pi}{\beta}\frac{6}{16}-\frac{2}{3}\frac{\varepsilon\pi}{\beta}\beta\gamma\frac{1}{16}\\ +\frac{\varepsilon\pi}{\beta}\alpha^4\frac{1}{4}+\frac{\varepsilon\pi}{\beta}\alpha^2\frac{1}{4}-\frac{\varepsilon\pi}{\beta}\alpha^4\frac{1}{8}-\frac{\varepsilon\pi}{\beta}\alpha^2\frac{1}{4}-\frac{2}{5}\frac{\varepsilon\pi}{\beta}\frac{5}{16}\end{array}\right)=$$

$$= A\frac{\varepsilon^2\pi}{\beta}\left(\frac{1}{8}+\frac{1}{4}\alpha^2+\frac{1}{8}\alpha^4-\beta\gamma\frac{\alpha^2}{8}-\frac{\beta\gamma}{24}\right) = A\frac{\varepsilon^2\pi}{\beta}\left(\frac{1}{8}+\frac{\alpha^2}{8}-\frac{\beta\gamma}{24}\right) = A\frac{\varepsilon^2\pi}{\beta}\frac{\beta\gamma}{12} = \frac{\varepsilon\gamma}{6}$$

$\therefore \varepsilon_{rms} = \sqrt{\left\langle x^2\right\rangle\left\langle x'^2\right\rangle - \left\langle xx'\right\rangle} = \varepsilon/6,$ and the expressions for $\alpha, \beta,$ and γ check out

This problem is far easier if the hint was used. The integrals become simple polar coordinates integrals that are much easier to evaluate.

$$\rho(x,x') = A\left(1-\left(\gamma x^2+2\alpha xx'+\beta x'^2\right)/\varepsilon\right)\Theta\left(1-\left(\gamma x^2+2\alpha xx'+\beta x'^2\right)/\varepsilon\right)$$

$$= A\left(1-\left(\left(1+\alpha^2\right)x^2+2\alpha\beta xx'+\beta^2x'^2\right)/\beta\varepsilon\right)\Theta\left(1-\left(\left(1+\alpha^2\right)x^2+2\alpha\beta xx'+\beta^2x'^2\right)/\beta\varepsilon\right)$$

$$= A\left(1-\frac{x^2}{\beta\varepsilon}-\frac{y^2}{\beta\varepsilon}\right)\Theta\left(1-\frac{x^2}{\beta\varepsilon}-\frac{y^2}{\beta\varepsilon}\right) \qquad \text{where } y = \alpha x + \beta x'$$

Note $dx' = dy/\beta$ for constant x

$$1 = A\iint \rho(x,x')dxdx' = \frac{A}{\beta}\iint\left(1-\frac{x^2}{\beta\varepsilon}-\frac{y^2}{\beta\varepsilon}\right)dydx$$

The region of integration is a circle, of radius $\sqrt{\varepsilon\beta}$ in xy space

$x = r\cos\theta$

$y = r\sin\theta$

$$1 = \frac{A}{\beta}\int_0^{2\pi}\int_0^{\sqrt{\beta\varepsilon}}\left(1-\frac{r^2}{\beta\varepsilon}\right)rdrd\theta = \frac{2\pi A}{\beta}\left[\frac{\beta\varepsilon}{2}-\frac{(\beta\varepsilon)^2}{4\beta\varepsilon}\right] = \frac{\pi\varepsilon A}{2} \to A = \frac{2}{\varepsilon\pi}$$

$$\left\langle x^2\right\rangle = \frac{A}{\beta}\int_0^{2\pi}\int_0^{\sqrt{\beta\varepsilon}} r^2\cos^2\theta\left(1-\frac{r^2}{\beta\varepsilon}\right)rdrd\theta = \frac{A\pi}{\beta}\left[\frac{(\beta\varepsilon)^2}{4}-\frac{(\beta\varepsilon)^3}{6\beta\varepsilon}\right]$$

$$= \frac{1}{12}\frac{A\pi}{\beta}(\beta\varepsilon)^2 = \frac{\beta\varepsilon}{6}$$

$$\left\langle xx'\right\rangle = \frac{A}{\beta}\int_0^{2\pi}\int_0^{\sqrt{\beta\varepsilon}} r\cos\theta\frac{(r\sin\theta-\alpha r\cos\theta)}{\beta}\left(1-\frac{r^2}{\beta\varepsilon}\right)rdrd\theta =$$

$$0-\frac{A\alpha}{\beta^2}\int_0^{2\pi}\int_0^{\sqrt{\beta\varepsilon}} r^2\cos^2\theta\left(1-\frac{r^2}{\beta\varepsilon}\right)rdrd\theta = -\frac{A\pi\alpha}{\beta^2}\left[\frac{(\beta\varepsilon)^2}{4}-\frac{(\beta\varepsilon)^3}{6\beta\varepsilon}\right]$$

$$= -\frac{1}{12}\frac{A\pi\alpha}{\beta^2}(\beta\varepsilon)^2 = -\frac{\alpha\varepsilon}{6}$$

$$\left\langle x'^2\right\rangle = \frac{A}{\beta}\int_0^{2\pi}\int_0^{\sqrt{\beta\varepsilon}}\frac{(r\sin\theta-\alpha r\cos\theta)^2}{\beta^2}\left(1-\frac{r^2}{\beta\varepsilon}\right)rdrd\theta =$$

$$\frac{A\alpha}{\beta^3}\int_0^{2\pi}\int_0^{\sqrt{\beta\varepsilon}}\left(r^2\sin^2\theta-2\alpha r^2\sin\theta\cos\theta+\alpha^2 r^2\cos^2\theta\right)\left(1-\frac{r^2}{\beta\varepsilon}\right)rdrd\theta =$$

$$\frac{A\pi}{\beta^3}\left(1-0+\alpha^2\right)\left[\frac{(\beta\varepsilon)^2}{4}-\frac{(\beta\varepsilon)^3}{6\beta\varepsilon}\right] = \frac{1}{12}\frac{A\pi}{\beta^3}\beta\gamma(\beta\varepsilon)^2 = \frac{\gamma\varepsilon}{6}$$

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