Mathematical Supplement
Electromagnetic Theory (Physics 704/804)

Some Math we'll need going forward

I will assume knowledge of vectors and vector fields

2-tensors: A 9-component object in 3-space (16 component object in 4-space) that transforms as "squares" of vectors in 3-space ("squares" of 4-vectors in 4-space).

Let $\vec{\mathrm{v}} = \mathrm{v}_x\hat{x} + \mathrm{v}_y\hat{y} + \mathrm{v}_z\hat{z}$ in 3-space

Form the object $\left(\vec{\mathrm{v}}\right)\left(\vec{\mathrm{v}}\right) = \left(\mathrm{v}_x\hat{x} + \mathrm{v}_y\hat{y} + \mathrm{v}_z\hat{z}\right)\left(\mathrm{v}_x\hat{x} + \mathrm{v}_y\hat{y} + \mathrm{v}_z\hat{z}\right) \equiv \mathrm{v}_x\mathrm{v}_x\hat{x}\hat{x} + \mathrm{v}_x\mathrm{v}_y\hat{x}\hat{y} + \cdots + \mathrm{v}_z\mathrm{v}_z\hat{z}\hat{z}$

and a correspondence between the squared object and 3 × 3 matrices (4 × 4 matrices) as

$$\mathrm{v}_x\mathrm{v}_x\hat{x}\hat{x} + \mathrm{v}_x\mathrm{v}_y\hat{x}\hat{y} + \cdots + \mathrm{v}_z\mathrm{v}_z\hat{z}\hat{z} \Leftrightarrow \begin{pmatrix} \mathrm{v}_x\mathrm{v}_x & \mathrm{v}_x\mathrm{v}_y & \mathrm{v}_x\mathrm{v}_z \\ \mathrm{v}_y\mathrm{v}_x & \mathrm{v}_y\mathrm{v}_y & \mathrm{v}_y\mathrm{v}_z \\ \mathrm{v}_z\mathrm{v}_x & \mathrm{v}_z\mathrm{v}_y & \mathrm{v}_z\mathrm{v}_z \end{pmatrix}.$$

Under a change of basis

$$\begin{gathered} \vec{e}_1 = E_{11}\hat{x} + E_{12}\hat{y} + E_{13}\hat{z} \\ \vec{e}_2 = E_{21}\hat{x} + E_{22}\hat{y} + E_{23}\hat{z} \\ \vec{e}_3 = E_{31}\hat{x} + E_{32}\hat{y} + E_{33}\hat{z} \\ E_{11} = \vec{e}_1 \cdot \hat{x}, E_{12} = \vec{e}_1 \cdot \hat{y}, \cdots, E_{33} = \vec{e}_3 \cdot \hat{z} \end{gathered}$$

with the matrix E nonsingular, to represent the same object, the vector components must be transformed by

$$\begin{pmatrix} \mathrm{v}'_1 \\ \mathrm{v}'_2 \\ \mathrm{v}'_3 \end{pmatrix} = E^{t-1}\begin{pmatrix} \mathrm{v}_x \\ \mathrm{v}_y \\ \mathrm{v}_z \end{pmatrix},$$

where *t* denotes the transpose, and the components of the squared quantity must be transformed as

$$\begin{pmatrix} \mathrm{v}'_x\mathrm{v}'_x & \mathrm{v}'_x\mathrm{v}'_y & \mathrm{v}'_x\mathrm{v}'_z \\ \mathrm{v}'_y\mathrm{v}'_x & \mathrm{v}'_y\mathrm{v}'_y & \mathrm{v}'_y\mathrm{v}'_z \\ \mathrm{v}'_z\mathrm{v}'_x & \mathrm{v}'_z\mathrm{v}'_y & \mathrm{v}'_z\mathrm{v}'_z \end{pmatrix} = E^{t-1}\begin{pmatrix} \mathrm{v}_x\mathrm{v}_x & \mathrm{v}_x\mathrm{v}_y & \mathrm{v}_x\mathrm{v}_z \\ \mathrm{v}_y\mathrm{v}_x & \mathrm{v}_y\mathrm{v}_y & \mathrm{v}_y\mathrm{v}_z \\ \mathrm{v}_z\mathrm{v}_x & \mathrm{v}_z\mathrm{v}_y & \mathrm{v}_z\mathrm{v}_z \end{pmatrix}E^{-1},$$

because then

$$\mathrm{v}_1'\mathrm{v}_1'\vec{e}_1\vec{e}_1 + \mathrm{v}_1'\mathrm{v}_2'\vec{e}_1\vec{e}_2 + \cdots + \mathrm{v}_3'\mathrm{v}_3'\vec{e}_3\vec{e}_3 = \mathrm{v}_x\mathrm{v}_x\hat{x}\hat{x} + \mathrm{v}_x\mathrm{v}_z\hat{x}\hat{y} + \cdots + \mathrm{v}_z\mathrm{v}_z\hat{z}\hat{z}\,.$$

Abstracting this idea, a 2-tensor is any 9-component (16-component) object whose components transform as the matrix equation above under a change of basis. This type of definition of tensors, in terms of transformation rules, is prevalent in the older literature, e.g., Einstein's papers on general relativity.

Going forward in the course, we shall generally write tensor components in terms of the standard $(\hat{x},\hat{y},\hat{z})$ basis or $(c\hat{t},\hat{x},\hat{y},\hat{z})$ "4-basis".

The squaring operation above is an example of a more general notion called the tensor product. As an example of this product: if $\vec{\mathrm{v}}_1 = \mathrm{v}_{1x}\hat{x} + \mathrm{v}_{1y}\hat{y} + \mathrm{v}_{1z}\hat{z}$ and $\vec{\mathrm{v}}_2 = \mathrm{v}_{2x}\hat{x} + \mathrm{v}_{2y}\hat{y} + \mathrm{v}_{2z}\hat{z}$ are vectors, then the quantity

$$T = \vec{\mathrm{v}}_1 \otimes \vec{\mathrm{v}}_2 \equiv \mathrm{v}_{1x}\mathrm{v}_{2x}\hat{x}\hat{x} + \mathrm{v}_{1x}\mathrm{v}_{2y}\hat{x}\hat{y} + \ldots + \mathrm{v}_{1z}\mathrm{v}_{2z}\hat{z}\hat{z}\,,$$

because the basis vectors and vector components transform exactly as in squaring above, is a nine component 2-tensor in 3-space. The tensor product, $\otimes$, which can be generalized beyond this specific application, provides and very powerful way to generate new, higher rank tensors from lower rank tensors. The tensor product is linear in each of its arguments.

$$\begin{aligned}
(\alpha\mathbf{u})\otimes\mathbf{v} &= \mathbf{u}\otimes(\alpha\mathbf{v}) = \alpha(\mathbf{u}\otimes\mathbf{v}),\\
\mathbf{u}\otimes(\mathbf{v}+\mathbf{w}) &= \mathbf{u}\otimes\mathbf{v} + \mathbf{u}\otimes\mathbf{w},\\
(\mathbf{u}+\mathbf{v})\otimes\mathbf{w} &= \mathbf{u}\otimes\mathbf{w} + \mathbf{v}\otimes\mathbf{w},\\
(\mathbf{u}\otimes\mathbf{v})\cdot\mathbf{w} &= \mathbf{u}\,(\mathbf{v}\cdot\mathbf{w}),\\
\mathbf{u}\cdot(\mathbf{v}\otimes\mathbf{w}) &= (\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}.
\end{aligned}$$

Note, by definition,

$$\hat{x}\hat{x} = \hat{x}\otimes\hat{x},\ \hat{x}\hat{y} = \hat{x}\otimes\hat{y},\ldots,\hat{z}\hat{z} = \hat{z}\otimes\hat{z}$$

and $\hat{x}\otimes\hat{y} \neq \hat{y}\otimes\hat{x}$.

Not all 2-tensors can be written $\vec{\mathrm{v}}_1\otimes\vec{\mathrm{v}}_2$ for some $\vec{\mathrm{v}}_1$ and $\vec{\mathrm{v}}_2$ (Exercise for reader: which ones can be so written?), but all can be written as "9-sums" (16-sums)

$$T = t_{11}\hat{x}\hat{x} + t_{12}\hat{x}\hat{y} + \cdots + t_{33}\hat{z}\hat{z}$$

for some real numbers t_{ij}, called the components of the tensor in the standard basis, where i and j extend from 1 to 3 (0 to 3). To prove this assertion note that if the 2-tensor is dotted into the standard basis vectors

$$t_{11} = \hat{x}\cdot T\cdot\hat{x},\ t_{12} = \hat{x}\cdot T\cdot\hat{y},\cdots,\ t_{33} = \hat{z}\cdot T\cdot\hat{z}\text{, then } T = t_{11}\hat{x}\hat{x} + t_{12}\hat{x}\hat{y} + \cdots + t_{33}\hat{z}\hat{z}\,.$$

The components are sometimes written in matrix form

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}\text{, or for 4-space } \begin{pmatrix} t_{00} & t_{01} & t_{02} & t_{03} \\ t_{10} & t_{11} & t_{12} & t_{13} \\ t_{20} & t_{21} & t_{22} & t_{23} \\ t_{30} & t_{31} & t_{32} & t_{33} \end{pmatrix},$$

and manipulated as a single entity.

Relationship between 3 by 3 matrices, 2-tensors, and bi-linear maps. Set up a correspondence

$$\hat{x}\hat{x} \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Leftrightarrow T_{\hat{x}\hat{x}}\left(\vec{\mathrm{v}}_1, \vec{\mathrm{v}}_2\right) = \left(\vec{\mathrm{v}}_1\cdot\hat{x}\right)\left(\hat{x}\cdot\vec{\mathrm{v}}_2\right) = \mathrm{v}_{1x}\mathrm{v}_{2x}$$

$$\hat{x}\hat{y} \Leftrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Leftrightarrow T_{\hat{x}\hat{y}}\left(\vec{\mathrm{v}}_1, \vec{\mathrm{v}}_2\right) = \left(\vec{\mathrm{v}}_1\cdot\hat{x}\right)\left(\hat{y}\cdot\vec{\mathrm{v}}_2\right) = \mathrm{v}_{1x}\mathrm{v}_{2y}$$

$$\vdots$$

$$\hat{z}\hat{z} \Leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Leftrightarrow T_{\hat{z}\hat{z}}\left(\vec{\mathrm{v}}_1, \vec{\mathrm{v}}_2\right) = \left(\vec{\mathrm{v}}_1\cdot\hat{z}\right)\left(\hat{z}\cdot\vec{\mathrm{v}}_2\right) = \mathrm{v}_{1z}\mathrm{v}_{2z}$$

Each of which is obviously bi-linear. Any bi-linear map $L : \mathrm{R}^3 \times \mathrm{R}^3 \to \mathrm{R}$, can be represented by a matrix, and hence by a corresponding tensor by expanding analogously to above. Let

$$l_{11} = L\left(\hat{x},\hat{x}\right),\ l_{12} = L\left(\hat{x},\hat{y}\right),\cdots,\ l_{33} = L\left(\hat{z},\hat{z}\right)\text{, then } L = l_{11}\hat{x}\hat{x} + l_{12}\hat{x}\hat{y} + \cdots + l_{33}\hat{z}\hat{z}$$

is the corresponding 2-tensor that gives the map by $L\left(\vec{\mathrm{v}}_1,\vec{\mathrm{v}}_2\right)=\vec{\mathrm{v}}_1\cdot L\cdot\vec{\mathrm{v}}_2$ (verify this assertion for arbitrary $\vec{\mathrm{v}}_1$ and $\vec{\mathrm{v}}_2$).

A general 2-tensor defines a bi-linear map $T:\mathrm{R}^3\times\mathrm{R}^3\to\mathrm{R},\quad T\left(\vec{\mathrm{v}}_1,\vec{\mathrm{v}}_2\right)=\vec{\mathrm{v}}_1\cdot T\cdot\vec{\mathrm{v}}_2$, which, by the tensor component transformation rule, does not depend on the basis in which it is evaluated (verify!). For the standard basis it evaluates to

$$T\left(\vec{\mathrm{v}}_1,\vec{\mathrm{v}}_2\right)=t_{11}\mathrm{v}_{1x}\mathrm{v}_{2x}+t_{12}\mathrm{v}_{1x}\mathrm{v}_{2y}+\cdots+t_{33}\mathrm{v}_{1z}\mathrm{v}_{2z}\,.$$

Note

$$\begin{aligned}T\left(\lambda\vec{\mathrm{v}}_1+\vec{\mathrm{v}}_2,\vec{\mathrm{v}}\right)&=t_{11}\left(\lambda\mathrm{v}_{1x}+\mathrm{v}_{2x}\right)\mathrm{v}_x+t_{12}\left(\lambda\mathrm{v}_{1x}+\mathrm{v}_{2x}\right)\mathrm{v}_y+\cdots+t_{33}\left(\lambda\mathrm{v}_{1z}+\mathrm{v}_{2z}\right)\mathrm{v}_z\\&=\lambda t_{11}\mathrm{v}_{1x}\mathrm{v}_x+t_{11}\mathrm{v}_{2x}\mathrm{v}_x+\lambda t_{12}\mathrm{v}_{1x}\mathrm{v}_y+t_{12}\mathrm{v}_{2x}\mathrm{v}_y+\cdots+\lambda t_{33}\mathrm{v}_{1z}\mathrm{v}_z+t_{33}\mathrm{v}_{2z}\mathrm{v}_z\\&=\lambda T\left(\vec{\mathrm{v}}_1,\vec{\mathrm{v}}\right)+T\left(\vec{\mathrm{v}}_2,\vec{\mathrm{v}}\right),\end{aligned}$$

and similarly $T\left(\vec{\mathrm{v}},\lambda\vec{\mathrm{v}}_1+\vec{\mathrm{v}}_2\right)=\lambda T\left(\vec{\mathrm{v}},\vec{\mathrm{v}}_1\right)+T\left(\vec{\mathrm{v}},\vec{\mathrm{v}}_2\right)$. So there is a direct one to one correspondence between 2-tensors and bi-linear maps. Some authors use this equivalence to *define* tensors very generally as multilinear maps on particular vector spaces. We will not need to employ this full generally.

A tensor is symmetric or antisymmetric depending on whether the representing matrix is symmetric or antisymmetric. It is straightforward to show, using the change of basis formula, that this is a frame-invariant notion.

Modern notation: Let the coordinates of R^3 be given by $x^1=x,x^2=y,x^3=z$. Following standard sloppy behavior not distinguishing the space R^3 from its tangent space, define the *x*-component operator by

$$dx^1\left(\vec{\mathrm{v}}\right)=dx\left(\vec{\mathrm{v}}\right)=\hat{x}\cdot\vec{\mathrm{v}}=\mathrm{v}_x\,.$$

This is a linear map in the tangent space to R^3, which following standard procedure is identified with R^3, and defined regardless of where in R^3 the vector is situated. Likewise define

$$\begin{aligned}dx^2\left(\vec{\mathrm{v}}\right)&=dy\left(\vec{\mathrm{v}}\right)=\hat{y}\cdot\vec{\mathrm{v}}=\mathrm{v}_y\\dx^3\left(\vec{\mathrm{v}}\right)&=dz\left(\vec{\mathrm{v}}\right)=\hat{z}\cdot\vec{\mathrm{v}}=\mathrm{v}_z.\end{aligned}$$

With the numbering convention above, it is clear (trace this through!) that the general 3-space (4-space) 2-tensor is given as

$$T = t_{ij} dx^i \otimes dx^j,$$

where i and j are summed from 1 to 3 (0 to 3). Going forward, the following Einstein summation will be followed. When an upper and lower latin index is used, the summation index goes from 1 to 3. When an upper and lower greek index is used, the sum proceeds from 0 (representing the time coordinate) to 3.

In 3-space, the anti-symmetrical tensors have an additional specialized notation

$$dx^2 \wedge dx^3 = dx^2 \otimes dx^3 - dx^3 \otimes dx^2 \Leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \Leftrightarrow dy \wedge dz\left(\vec{\mathrm{v}}_1, \vec{\mathrm{v}}_2\right) = \mathrm{v}_{1y}\mathrm{v}_{2z} - \mathrm{v}_{1z}\mathrm{v}_{2y}$$

$$dx^3 \wedge dx^1 = dx^3 \otimes dx^1 - dx^1 \otimes dx^3 \Leftrightarrow \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \Leftrightarrow dz \wedge dx\left(\vec{\mathrm{v}}_1, \vec{\mathrm{v}}_2\right) = \mathrm{v}_{1z}\mathrm{v}_{2x} - \mathrm{v}_{1x}\mathrm{v}_{2z}$$

$$dx^1 \wedge dx^2 = dx^1 \otimes dx^2 - dx^2 \otimes dx^1 \Leftrightarrow \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Leftrightarrow dx \wedge dy\left(\vec{\mathrm{v}}_1, \vec{\mathrm{v}}_2\right) = \mathrm{v}_{1x}\mathrm{v}_{2y} - \mathrm{v}_{1y}\mathrm{v}_{2x}$$

These formulas should remind one of the projection of the vector cross product (with proper sign) onto the missing coordinate axes. Also, one can give a more physical interpretation of the antisymmetric product . $dx \wedge dy$ operating on the pair $\left(\vec{\mathrm{v}}_1, \vec{\mathrm{v}}_2\right)$ gives the area of the parallelepiped whose edges are formed by the projection of the two vectors into the *x-y* plane. Likewise $dy \wedge dz$ gives the area of the parallelepiped whose edges are formed by the projection of the two vectors into the *y-z* plane, and $dz \wedge dx$ the area after projection into the *z-x* plane. The order is important, positive means the projection of $\vec{\mathrm{v}}_1$, the projection of $\vec{\mathrm{v}}_2$, and the remaining positive axis direction form a right handed set of vectors. Also clearly, any antisymmetric tensor can be written as the sum of these three basis tensors. For four space, 2- tensors have 16 components and the antisymmetic tensors have up to six independent components. In this case, the basic antisymmetric tensors are

$$cdt \wedge dx^1 = cdt \otimes dx^1 - cdx^1 \otimes dt \Leftrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$cdt \wedge dx^2 = cdt \otimes dx^2 - cdx^2 \otimes dt \Leftrightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$cdt \wedge dx^3 = cdt \otimes dx^3 - cdx^3 \otimes dt \Leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$dx^1 \wedge dx^2 = dx^1 \otimes dx^2 - dx^2 \otimes dx^1 \Leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$dx^2 \wedge dx^3 = dx^2 \otimes dx^3 - dx^3 \otimes dx^2 \Leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$dx^3 \wedge dx^1 = dx^3 \otimes dx^1 - dx^1 \otimes dx^3 \Leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Differential Forms

In this course we'll restrict ourselves to differential forms on flat 3-space, or on special relativity's Minkowski space. The same general constructions can be used, with even more power, on general manifolds and the curved spacetimes of general relativity. These complications are not useful in a course on electrodynamics. Also, we will generally assume that functions have "as many derivatives as necessary" to ensure derivations are valid.

A differentiable function $f:\mathrm{R}^3 \to \mathrm{R}$ will be called a 0-form.

A 1-form is any mapping, linear on vector spaces, of the form

$$\begin{aligned}\omega^1\left(\vec{\mathrm{v}}(x,y,z)\right) = a_1(x,y,z)\,dx^1\left(\vec{\mathrm{v}}(x,y,z)\right) + a_2(x,y,z)\,dx^2\left(\vec{\mathrm{v}}(x,y,z)\right) \\ + a_3(x,y,z)\,dx^3\left(\vec{\mathrm{v}}(x,y,z)\right),\end{aligned},$$

where the a_i are differentiable functions, and as above, the dx^i project out specific components of the vector field at the location in question. 1-forms operate on vector fields and yield a scalar (i.e., basis independent) 0-form on evaluation with a specific vector field.

A 2-form is the analogous construction applied to antisymmetric 2-tensor fields. It is a bi-linear anti-symmetric mapping of the form

$$\begin{aligned}\omega^2\left(\vec{\mathrm{v}}_1,\vec{\mathrm{v}}_2\right) = a_1 dx^2 \wedge dx^3\left(\vec{\mathrm{v}}_1,\vec{\mathrm{v}}_2\right) + a_2 dx^3 \wedge dx^1\left(\vec{\mathrm{v}}_1,\vec{\mathrm{v}}_2\right) \\ + a_3 dx^1 \wedge dx^2\left(\vec{\mathrm{v}}_1,\vec{\mathrm{v}}_2\right),\end{aligned}$$

where, for convenience of notation, the (x,y,z) dependence of the a_i and the two vector fields is suppressed. It was noted above that any antisymmetric 2-tensor field must be of this form.

A 3-form, or totally antisymmetic 3-tensor field, is given as

$$\begin{aligned}\omega^3\left(\vec{\mathrm{v}}_1,\vec{\mathrm{v}}_2,\vec{\mathrm{v}}_3\right) = a dx^1 \wedge dx^2 \wedge dx^3\left(\vec{\mathrm{v}}_1,\vec{\mathrm{v}}_2,\vec{\mathrm{v}}_3\right) = \\ a\begin{pmatrix} dx^1 \otimes dx^2 \otimes dx^3 + dx^2 \otimes dx^3 \otimes dx^1 + dx^3 \otimes dx^1 \otimes dx^2 \\ -dx^1 \otimes dx^3 \otimes dx^2 - dx^2 \otimes dx^1 \otimes dx^3 - dx^3 \otimes dx^2 \otimes dx^1 \end{pmatrix}\left(\vec{\mathrm{v}}_1,\vec{\mathrm{v}}_2,\vec{\mathrm{v}}_3\right)\end{aligned}$$

where a is a differentiable function. When evaluated on the triple $\left(\vec{\mathrm{v}}_1,\vec{\mathrm{v}}_2,\vec{\mathrm{v}}_3\right)$, ω^3 / a yields the so-called triple scalar product of the vectors $\vec{\mathrm{v}}_1 \cdot \vec{\mathrm{v}}_2 \times \vec{\mathrm{v}}_3$, which is the oriented volume of the parallelliped with sides given by the three vectors. There are no higher forms in 3-space. It is an exercise to write out the corresponding forms, including the possibility of 4-forms, in Minkowski space.

Exterior Derivative Map d

In this document, the exterior derivative will be defined operationally. Refer to any good book on differential forms for more complete derivations. The exterior product always carries an i-form to an $i+1$-form and is evaluated using several basic rules

1. $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$ when evaluated on a zero-form. The linear map $df\left(\vec{\mathrm{v}}\right)$ is better known as the directional derivative of the function f in the direction $\vec{\mathrm{v}}$. Note that when

evaluated on the coordinate functions, it evaluates to the correct value. For example $dx = 1 \cdot dx + 0 \cdot dy + 0 \cdot dz$.

2. $d\left(a \wedge b\right) = da \wedge b + \left(-1\right)^i a \wedge db$ where a is an i-form, the modified Leibnitz rule
3. The antisymmetric product $\wedge$ is anti-commutative on 1-forms (show this!!) and thus $dx \wedge dx = dy \wedge dy = dz \wedge dz = 0$
4. $d^2\omega \equiv d\left(d\omega\right) = 0$ for any form because mixed partial derivatives are always equal

Examples: Suppose the general 1-form is written in the following suggestive way

$$\omega^1 = a_1 dx^1 + a_2 dx^2 + a_3 dx^3.$$

Then

$$\begin{aligned} d\omega^1 = &\left(\frac{\partial a_1}{\partial x^1} dx^1 + \frac{\partial a_1}{\partial x^2} dx^2 + \frac{\partial a_1}{\partial x^3} dx^3\right) \wedge dx^1 + 0 \\ &+ \left(\frac{\partial a_2}{\partial x^1} dx^1 + \frac{\partial a_2}{\partial x^2} dx^2 + \frac{\partial a_2}{\partial x^3} dx^3\right) \wedge dx^2 + 0 \\ &+ \left(\frac{\partial a_3}{\partial x^1} dx^1 + \frac{\partial a_3}{\partial x^2} dx^2 + \frac{\partial a_3}{\partial x^3} dx^3\right) \wedge dx^3 + 0 = \\ &\left(\frac{\partial a_3}{\partial x^2} - \frac{\partial a_2}{\partial x^3}\right) dx^2 \wedge dx^3 + \left(\frac{\partial a_1}{\partial x^3} - \frac{\partial a_3}{\partial x^1}\right) dx^3 \wedge dx^1 + \left(\frac{\partial a_2}{\partial x^1} - \frac{\partial a_1}{\partial x^2}\right) dx^1 \wedge dx^2. \end{aligned}$$

Likewise if

$$\omega^2 = a_1 dx^2 \wedge dx^3 + a_2 dx^3 \wedge dx^1 + a_3 dx^1 \wedge dx^2,$$

$$\begin{aligned} d\omega^2 = &\left(\frac{\partial a_1}{\partial x^1} dx^1 + \frac{\partial a_1}{\partial x^2} dx^2 + \frac{\partial a_1}{\partial x^3} dx^3\right) \wedge dx^2 \wedge dx^3 + 0 \\ &+ \left(\frac{\partial a_2}{\partial x^1} dx^1 + \frac{\partial a_2}{\partial x^2} dx^2 + \frac{\partial a_2}{\partial x^3} dx^3\right) \wedge dx^3 \wedge dx^1 + 0 \\ &+ \left(\frac{\partial a_3}{\partial x^1} dx^1 + \frac{\partial a_3}{\partial x^2} dx^2 + \frac{\partial a_3}{\partial x^3} dx^3\right) \wedge dx^1 \wedge dx^2 + 0 = \\ &\left(\frac{\partial a_1}{\partial x^1} + \frac{\partial a_2}{\partial x^2} + \frac{\partial a_3}{\partial x^3}\right) dx^1 \wedge dx^2 \wedge dx^3 \end{aligned}$$

If a regular vector is associated with a 1-form

$$\omega_{\mathrm{v}}^1 = \mathrm{v}_i dx^i,$$

d generates the (polar vector!) curl of the vector field. If a polar vector is associated with a 2-form

$$\omega_{\mathrm{v}}^{2} = \varepsilon_{ijk}\mathrm{v}_i dx^j \otimes dx^k,$$

d generates the divergence of the starting vector field. Therefore, the standard gradient, divergence, and curl operations in vector analysis are included as cases of the d operations in forms. We will take the approach that the usual vector analysis is most naturally analyzed with mathematics of forms. The following correspondences apply

$$\begin{array}{ccc}
\{0\text{-}forms\} & \leftrightarrow & \{functions\} \\
d\downarrow & & \downarrow\ grad \\
\{1\text{-}forms\} & \leftrightarrow & \{vector\ fields\} \\
d\downarrow & & \downarrow\ curl \\
\{2\text{-}forms\} & \leftrightarrow & \{polar\ vector\ fields\} \\
d\downarrow & & \downarrow\ div \\
\{3\text{-}forms\} & \leftrightarrow & \{functions\}
\end{array}$$

The fact that $d^2 = 0$ is equivalent to the vector analysis statements that $\nabla \times (\nabla f) = 0$ and $\nabla \cdot (\nabla \times \vec{\mathrm{v}}) = 0$.

Forms such that $d\omega = 0$ are called *closed.* Forms ω that can be written as the exterior derivative of another, lower degree, form $\omega = da$, are called *exact*. It is a mathematical result, called the Poincaré Lemma, that because R3 is simply connected (contractible is enough), all closed forms are exact in 3-space. The same result applies in Minkowski space. In E&M theory this fact is used constantly, particularly when the scalar potential and vector potential functions are introduced to represent electromagnetic fields.

Integration and generalized Stokes Theorem.

0-forms cannot be integrated, only evaluated at specific locations.

1-forms are integrated as line integrals. Let C be a curve, open or closed parameterized by an independent variable t. Then

$$\int_C \omega^1 \equiv \int_C \left(a_1\left(\vec{x}(t)\right)\frac{dx}{dt} + a_2\left(\vec{x}(t)\right)\frac{dy}{dt} + a_3\left(\vec{x}(t)\right)\frac{dz}{dt} \right) dt.$$

For exact forms $\omega^1 = d\phi$, the integral evaluates as the difference of the function value $\phi\left(\vec{x}(t')\right) - \phi\left(\vec{x}(t)\right)$ at the endpoints of the curve, $\vec{x}(t')$ and $\vec{x}(t')$. This is the case of a conservative force field with ϕ the potential function. For a closed curve the line integral evaluates to zero for exact 1-forms.

2-forms are integrated as surface integrals. Let S be a surface written, for example, as $\left(x, y, z = s\left(x, y\right)\right)$. Now $dz = \frac{\partial s}{\partial x} dx + \frac{\partial s}{\partial y} dy$, so

$$\int_S \omega^2 \equiv \int_S \left(a_1\left(x, y, s\left(x, y\right)\right) dy \wedge \frac{\partial s}{\partial x} dx + a_2\left(x, y, s\left(x, y\right)\right) \frac{\partial s}{\partial y} dy \wedge dx + a_3\left(x, y, s\left(x, y\right)\right) dx \wedge dy \right)$$
$$= \int_{p(S)} \left(-a_1\left(x, y, s\left(x, y\right)\right) \frac{\partial s}{\partial x} - a_2\left(x, y, s\left(x, y\right)\right) \frac{\partial s}{\partial y} + a_3\left(x, y, s\left(x, y\right)\right) \right) dxdy$$

where $p\left(S\right)$ is the projected area of the surface into the x - y plane and the final integral is the usual 2-D multiple integral. Note that for the integral to make sense, s must be single-valued within the integration domain $p\left(S\right)$. Total surface integrals must be broken up into sums of single-valued portions in cases, such as when the surface is closed, where z is not uniquely determined by x and y. It also may be convenient, and is possible, to express some portions of the surface using the independent variables x and z or y and z, or even using parametric representations of the surface, in order to evaluate parts of, or the complete surface integral.

Now $\vec{n} = \left(-\partial z / \partial x, -\partial z / \partial y, 1\right)$ is in the direction of the normal to the surface. So

$$\int_S \omega^2 = \int_{p(S)} \left(\vec{a} \cdot \hat{n}\right) \left|\vec{n}\right| dxdy = \int_S \left(\vec{a} \cdot \hat{n}\right) dA$$

where dA is the local area element of the surface, that is projected down into the area element $dxdy$ by the magnitude of $\vec{n}$. This final form of the integral is that usually found in vector analysis.

Flux integrals are performed by doing such surface integrals. The total surface integral evaluates to zero on a closed surface (i.e., a surface with no boundary) if the 2-form is exact.

3-forms are integrated as signed volume integrals

$$\int_V \omega^3 \equiv \int_V a\left(x, y, z\right) dx \wedge dy \wedge dz = \int_V a\left(x, y, z\right) dxdydz.$$

where the final integral is the usual 3-d multiple integral.

Generalized Stokes Theorem (for a proof, and conditions for validity, see any good book on forms)

If M is a manifold and ω a form that exists throughout the manifold, then

$$\int_M d\omega = \int_{\partial M} \omega$$

where ∂M is the boundary of M . We will be applying this theorem to various regions of 3-space and Minkowski space, although it works for, and inside of much more general manifolds. Within it are the main theorems of vector analysis.

Fundamental Theorem of Calculus: Applied to [a,b] in R . If $\omega = f$

$$\int_{[a,b]} \frac{df}{dx} dx = f\Big|_a^b = f(b) - f(a)$$

Green's Theorem: Applied to a simply connected region R of R^2 . If $\omega = Pdx + Qdy$

$$\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \int_{\partial R} Pdx + Qdy = \int_{p(\partial R)} (P + Qdy/dx) dx.$$

The second equality applies in the case that $y(x)$ is the monotonic, and single-valued equation for the boundary curve in terms of the independent variable x. Note that if $d\omega = 0$, i.e. the form is exact, the line integrals are always independent of path, as above.

Stoke's Theorem: Applied to a simply connected surface in R^3. If $\omega = Pdx + Qdy + Rdz$ and $\vec{A}(x,y,z) \equiv (P(x,y,z), Q(x,y,z), R(x,y,z))$

$$\int_S (\nabla \times \vec{A}) \cdot \hat{n} dS = \int_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$
$$= \int_{\partial S} Pdx + Qdy + Rdz = \oint_{\partial S} \vec{A} \cdot d\vec{r}.$$

.

Note that if ω is exact, $\vec{A} = \nabla \phi$ for some function, and we again have independence of path. In this case all surface integrals vanish, consistent with the usual vector analysis result that $\nabla \times \nabla \phi = 0$. A non-zero line integral requires a curl to be generated.

Divergence, or Gauss's Theorem: Applied to a simply connected volume in R^3. If $\omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$, $\vec{A}(x,y,z) \equiv (P(x,y,z), Q(x,y,z), R(x,y,z))$

$$\int_V (\nabla \cdot \vec{A}) dV = \int_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx \wedge dy \wedge dz = \int_{\partial V} Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy = \int_{\partial V} \vec{A} \cdot \hat{n} dS.$$

If ω is exact, i.e., $\nabla \cdot \vec{A} = 0$, then suppose c is a closed curve in boundary V. The flux integral through the curve is independent of the bounding surface used to evaluate the flux . This is a 2-dimensional result analogous to the path independent integrals for 1-forms. This result has implications for magnetic field lines, which always must close as $\nabla \cdot \vec{B} = 0$.

Extra formulas in Jackson: Let $Q = R = 0$ in the Divergence Theorem, and $P = \psi$. Then

$$\int_V \frac{\partial \psi}{\partial x} dx \wedge dy \wedge dz = \int_{\partial V} \psi dy \wedge dz = \int_{\partial V} \psi \left(\hat{n} \cdot \hat{x}\right) dS.$$

Likewise,

$$\int_V \frac{\partial \psi}{\partial y} dx \wedge dy \wedge dz = \int_{\partial V} \psi dz \wedge dx = \int_{\partial V} \psi \left(\hat{n} \cdot \hat{y}\right) dS \quad \text{and} \quad \int_V \frac{\partial \psi}{\partial z} dx \wedge dy \wedge dz = \int_{\partial V} \psi dx \wedge dy = \int_{\partial V} \psi \left(\hat{n} \cdot \hat{z}\right) dS.$$

Summing with the constant unit vectors gives

$$\int_V \nabla \psi dxdydz = \int_{\partial V} \psi \hat{n} dS$$

Using formulas like

$$\int_V \frac{\partial A_y}{\partial x} dx \wedge dy \wedge dz = \int_{\partial V} A_y dz \wedge dx = \int_{\partial V} A_y \left(\hat{n} \cdot \hat{x}\right) dS$$

One readily verifies that

$$\int_V \nabla \times \vec{A} dxdydz = \int_{\partial V} \hat{n} \times \vec{A} dS.$$

Following a similar analysis, let $Q = R = 0$ in Stokes Theorem, and $P = \psi$. Then

$$\int_S \frac{\partial \psi}{\partial z} \left(\hat{n} \cdot \hat{y}\right) dS - \frac{\partial \psi}{\partial y} \left(\hat{n} \cdot \hat{z}\right) dS = \int_S \frac{\partial \psi}{\partial z} dz \wedge dx - \frac{\partial \psi}{\partial y} dx \wedge dy = \int_{\partial S} \psi dx.$$

Likewise,

$$\int_S -\frac{\partial \psi}{\partial z} \left(\hat{n} \cdot \hat{x}\right) dS + \frac{\partial \psi}{\partial x} \left(\hat{n} \cdot \hat{z}\right) dS = \int_{\partial S} \psi dy, \quad \text{and} \quad \int_S \frac{\partial \psi}{\partial y} \left(\hat{n} \cdot \hat{x}\right) dS - \frac{\partial \psi}{\partial x} \left(\hat{n} \cdot \hat{y}\right) dS = \int_{\partial S} \psi dz.$$

Summing, with unit vectors as before yields

$$\int_S \hat{n} \times \nabla \psi dS = \int_{\partial S} \psi d\vec{r}.$$

This document provides a condensed summary of the mathematics of forms that we will need going forward. We will be expressing the electromagnetic field quantities in terms of forms; applying these methods will allow one to derive fairly deep results more quickly and rigorously than is possible using more standard vector analysis. One motivation for this adoption is the idea that "forms are made to be integrated". 1-forms are appropriately evaluated as line integrals, 2-forms are evaluated as surface integrals, and 3-forms as volume integrals. One can work this backwards. If one has a physical quantity best evaluated with a line integral, a 1-form is the appropriate description. Flux integrals must be completed with a 2-form. Quantities that involve volume densities, for example the charge density, are best represented as 3-forms.