Accelerator Physics
Xray Sources and FELs

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Lecture 13

High Field Spectral Distribution

In the beam frame

$$\frac{dE'_{perp,n}}{d\omega' d\Omega'} = \frac{e^2 \omega'^2_0}{2\pi^2 c} n^2 \Gamma^2_{xn} \sin^2 \phi' \sigma'^2_n \left(\omega'; \omega'_0\right)$$

(2.16)

$$\frac{dE'_{par,n}}{d\omega' d\Omega'} = \frac{e^2 \omega'^2_0}{2\pi^2 c} n^2 \left[\begin{array}{c} S_{1n} \dfrac{\cos\theta'}{\sin\theta'} + \\ \dfrac{S_{2n}}{n} \left(\dfrac{\cos\theta'}{\sin\theta'} + \dfrac{\sin\theta'}{\left(\beta^*_z + \cos\theta'\right)} \right) \end{array} \right]^2 \sigma'^2_n \left(\omega'; \omega'_0\right)$$

where

$$\sigma'_n \left(\omega'; \omega'_0\right) = f_{nN} \left(\omega'; n\omega'_0\right) f_1 \left(\omega'; n\omega'_0\right) \approx \frac{\sin\left(\pi n N \omega' / n\omega'_0\right)}{\sin\left(\pi \omega' / n\omega'_0\right)} \frac{\pi}{n\omega'_0}$$

In the lab frame

$$\frac{dE_{perp,n}}{d\omega d\Omega}=\frac{e^2}{2c}\frac{1}{\gamma *^2 \left(1-\beta *_z \cos\theta\right)^2}\frac{\gamma *^2 \left(1-\beta *_z \cos\theta\right)^2}{\sin^2\theta\cos^2\phi}$$

$$\cdot\begin{bmatrix} S_{1n} + \\ S_{2n}/n \end{bmatrix}^2 \sin^2\phi f_{nN}^2\left(\omega;n\omega(\theta)\right)$$

$$\frac{dE_{par,n}}{d\omega d\Omega}=\frac{e^2}{2c}\frac{1}{\gamma *^2 \left(1-\beta *_z \cos\theta\right)^2}\begin{bmatrix} S_{1n}\dfrac{\gamma *\left(\cos\theta-\beta *_z\right)}{\sin\theta} + \\ \dfrac{S_{2n}}{n}\dfrac{\gamma *\left(1-\beta *_z \cos\theta\right)}{\sin\theta\cos\theta} \end{bmatrix}^2 f_{nN}^2\left(\omega;n\omega(\theta)\right)$$

$$f_{nN}\left(\omega;n\omega(\theta)\right)\approx\frac{\sin\left(\pi nN\omega\left(1-\beta *_z \cos\theta\right)/\beta *_z n\omega_0\right)}{\sin\left(\pi\omega\left(1-\beta *_z \cos\theta\right)/\beta *_z n\omega_0\right)}$$

$$\frac{dE_{perp,n}}{d\omega d\Omega}=\frac{e^2}{2c}\left[S_{1n}+S_{2n}/n\right]^2\frac{\sin^2\phi}{\sin^2\theta\cos^2\phi}f_{nN}^2(\omega;n\omega(\theta))$$

$$\frac{dE_{par,n}}{d\omega d\Omega}=\frac{e^2}{2c}\left[\begin{array}{c}\dfrac{S_{1n}\left(\cos\theta-\beta^*_z\right)}{\left(1-\beta^*_z\cos\theta\right)\sin\theta}\\ +\dfrac{S_{2n}}{n\sin\theta\cos\theta}\end{array}\right]^2 f_{nN}^2(\omega;n\omega(\theta))$$

f_{nN} is highly peaked, with peak value nN, around angular frequency

$$n\omega(\theta)=\frac{\beta^*_z n\omega_0}{\left(1-\beta^*_z\cos\theta\right)}\to 2\gamma^{*2}\beta^*_z n\omega_0\approx\frac{2\gamma^2}{1+K^2/2}n\omega_0 \quad \text{as} \quad \theta\to 0$$

Energy Distribution in Lab Frame

$$\frac{dE_{perp,n}}{d\omega d\Omega} = \frac{e^2}{2c}\left[S_{1n} + S_{2n}/n\right]^2 \frac{\sin^2\phi}{\sin^2\theta\cos^2\phi} f_{nN}^2\left(\omega; n\omega(\theta)\right)$$

$$\frac{dE_{par,n}}{d\omega d\Omega} = \frac{e^2}{2c}\left[\begin{array}{c}\dfrac{S_{1n}\left(\cos\theta - \beta^*_z\right)}{\left(1-\beta^*_z\cos\theta\right)\sin\theta} \\ +\dfrac{S_{2n}}{n\sin\theta\cos\theta}\end{array}\right]^2 f_{nN}^2\left(\omega; n\omega(\theta)\right) \qquad (2.17)$$

The arguments of the Bessel Functions are now

$$\xi_x \equiv n\sin\theta'\cos\phi' d_x\omega'_0/c = n\frac{\sin\theta\cos\phi}{\left(1-\beta^*_z\cos\theta\right)}\frac{K}{\gamma}$$

$$\xi_z \equiv n\left(\beta^*_z + \cos\theta'\right)d_z\omega'_0/c = n\frac{\cos\theta}{\left(1-\beta^*_z\cos\theta\right)}\frac{\beta^*_z K^2}{8\gamma^2\beta^2}$$

In the Forward Direction

In the forward direction even harmonics vanish ($n+2k'$ term vanishes when "x" Bessel function non-zero at zero argument, and all other terms in sum vanish with a power higher than 2 as the argument goes to zero), and for odd harmonics only $n+2k'=1,-1$ contribute to the sum

$$\frac{dE_{perp,n}}{d\omega d\Omega} = \frac{e^2}{2c}\gamma^2\left(\frac{F_n(K)}{n^2}\right)\sin^2\phi\; f_{nN}^2\left(\omega; n\omega(\theta=0)\right)$$

$$\frac{dE_{par,n}}{d\omega d\Omega} = \frac{e^2}{2c}\gamma^2\left(\frac{F_n(K)}{n^2}\right)\cos^2\phi\; f_{nN}^2\left(\omega; n\omega(\theta=0)\right)$$

$$F_n(K) \approx \frac{1}{\gamma^2}\frac{n^2}{4\left(1-\beta^*_z\right)^2}\frac{K^2}{\gamma^2}\left[J_{\frac{n-1}{2}}\left(\frac{nK^2}{4\left(1+K^2/2\right)}\right) - J_{\frac{n+1}{2}}\left(\frac{nK^2}{4\left(1+K^2/2\right)}\right)\right]^2$$

Number Spectral Angular Density

Converting the energy density into an number density by dividing by the photon energy (don't forget both signs of frequency!)

$$\frac{dN_{perp,n}}{(d\omega/\omega)d\Omega} = \alpha\gamma^2 \left(\frac{F_n(K)}{n^2}\right) \sin^2\phi \; f_{nN}^2\left(\omega; n\omega(\theta=0)\right)$$

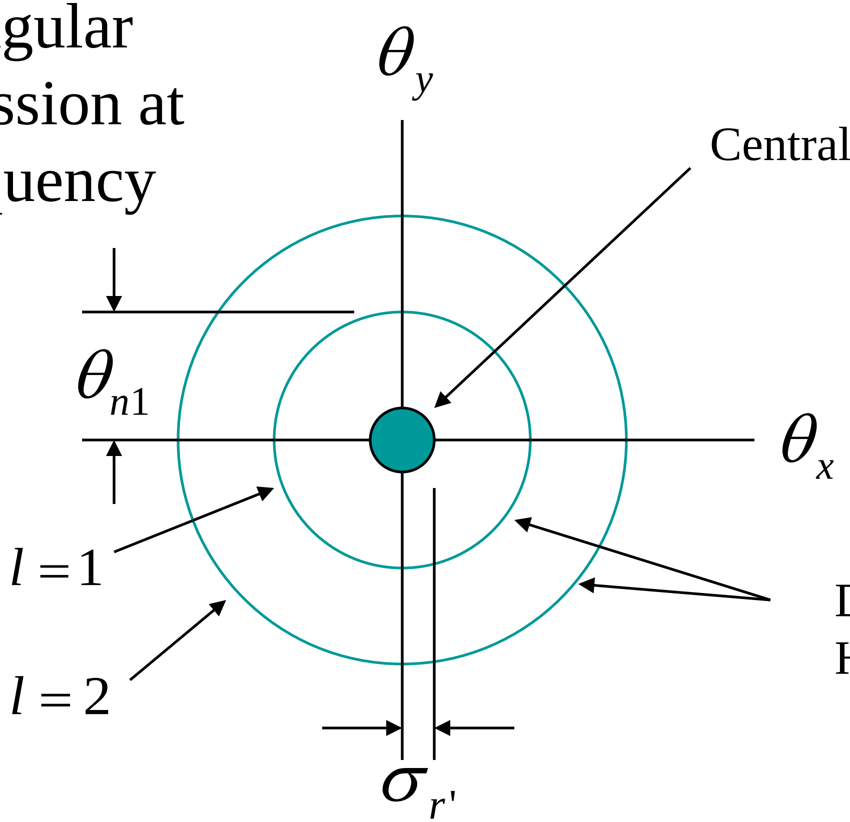
$$\frac{dN_{par,n}}{(d\omega/\omega)d\Omega} = \alpha\gamma^2 \left(\frac{F_n(K)}{n^2}\right) \cos^2\phi \; f_{nN}^2\left(\omega; n\omega(\theta=0)\right)$$

Peak value in the forward direction

$$\frac{dN_{tot,n}}{(d\omega/\omega)d\Omega} = \alpha\gamma^2 N^2 F_n(K)$$

Radiation Pattern: Qualitatively

- Non-zero Angular Density Emission at a Given Frequency



Central cone: high angular density region around forward direction

Dimension Estimates

Harmonic bands at

$$\theta_{nl} = \frac{1}{\gamma}\sqrt{\frac{l}{n}\left(1 + K^2/2\right)}$$

Central cone size estimated by requiring Gaussian distribution with correct peak value integrate over solid angle to the same number of total photons as integrating *f*

$$\sigma_{r'} = \frac{1}{2\gamma}\sqrt{\frac{\left(1 + K^2/2\right)}{nN}} = \sqrt{\frac{\lambda_n}{2L}} \qquad \lambda_n = c/n\omega(\theta = 0)$$

Much narrower than typical opening angle for bend

Number Spectral Density (Flux)

The flux in the central cone is obtained by estimating solid angle integral by the peak angular density multiplied by the Gaussian integral

$$\mathrm{F}^{\mathrm{n}} = \left.\frac{dN_{tot,n}}{d\Omega}\right|_{\theta=0} 2\pi\sigma_{r'}^{2}$$

$$\mathrm{F}^{\mathrm{n}} = \pi\alpha N \frac{\Delta\omega}{\omega}\frac{I}{e} g_n(K)$$

$$g_n(K) = \left(1 + K^2/2\right) F_n(K)/n$$

Power Angular Density

$$\frac{dE_{perp,n}}{d\Omega} = \alpha N n \hbar \omega(\theta) [S_{1n} + S_{2n} / n]^2 \frac{\sin^2\phi}{\sin^2\theta \cos^2\phi}$$

$$\frac{dE_{par,n}}{d\Omega} = \alpha N n \hbar \omega(\theta) \left[\frac{S_{1n}(\cos\theta - \beta^*_z)}{(1-\beta^*_z \cos\theta)\sin\theta} + \frac{S_{2n}}{n \sin\theta \cos\theta} \right]^2$$

Don’t forget both signs of frequency!

For K less than or of order one

$$\frac{dN_{n,perp}}{d\Omega} \approx \frac{\alpha}{4} \frac{NF_n(K)}{\gamma^2 \left(1 - \beta^*_z \cos\theta\right)^2} \sin^2\phi$$

$$\frac{dN_{n,par}}{d\Omega} \approx \frac{\alpha}{4} \frac{NF_n(K)}{\gamma^2 \left(1 - \beta^*_z \cos\theta\right)^2} \left(\frac{\cos\theta - \beta^*_z}{1 - \beta^*_z \cos\theta} \right)^2 \cos^2\phi$$

$$F_n(K) = \frac{n^2 K^2}{\left(1 + K^2/2\right)^2} \left\{ J_{\frac{n-1}{2}}\left(\frac{nK^2}{4\left(1 + K^2/2\right)} \right) - J_{\frac{n+1}{2}}\left(\frac{nK^2}{4\left(1 + K^2/2\right)} \right) \right\}^2$$

Compare with (2.10)

?

ERL light source idea

Third generation light sources are storage ring based facilities optimized for production of high brilliance x-rays through spontaneous synchrotron radiation. The technology is mature, and while some improvement in the future is likely, one ought to ask whether an alternative approach exists.

Two orthogonal ideas (both linac based) are XFEL and ERL. XFEL will not be spontaneous synchrotron radiation source, but will deliver GW peak powers of transversely coherent radiation at very low duty factor. The source parameters are very interesting and at the same time very different from any existing light source.

ERL aspires to do better what storage rings are very good at: to provide radiation in quasi-continuous fashion with superior brilliance, monochromaticity and shorter pulses.

Coherent or incoherent?

Radiation field from a single k^{th} electron in a bunch:

$$E_k = E_0 \exp(i\omega t_k)$$

Radiation field from the whole bunch $\propto$ bunching factor (*b.f.)*

$$b.f. = \frac{1}{N_e}\sum_{k=1}^{N_e}\exp(i\omega t_k)$$

Radiation Intensity:

$$I = I_0 \left|b.f.\right|^2 N_e^2$$

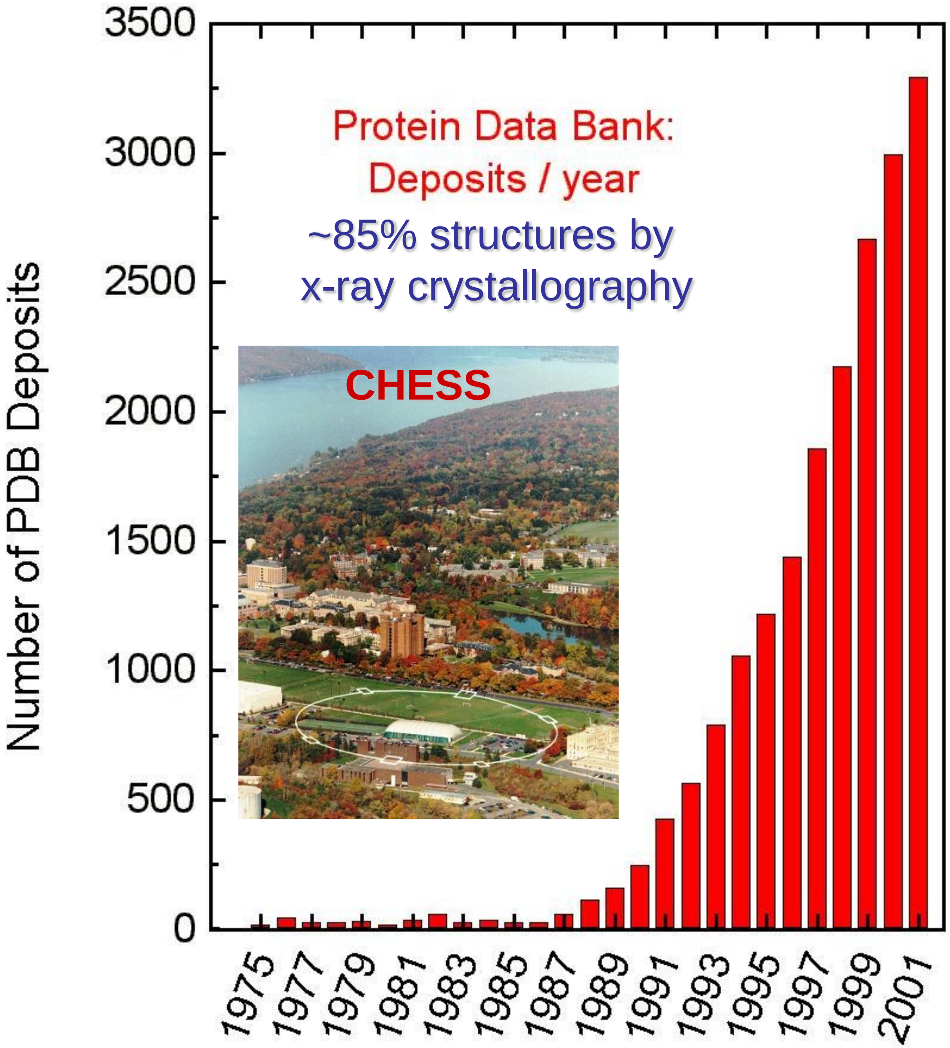
single electron

1) "long bunch": $\left|b.f.\right|^2 \sim 1/N_e$ => $I = I_0 N_e$ *incoherent (conventional) SR*

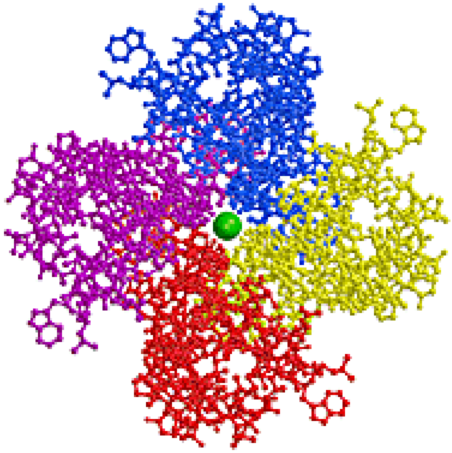
2) "short bunch" or μ-bunching: $\left|b.f.\right| \leq 1$ => $I \sim I_0 N_e^2$ *coherent (FELs) SR*

ERL hard x-ray source is envisioned to use conventional SR

Demand for X-rays



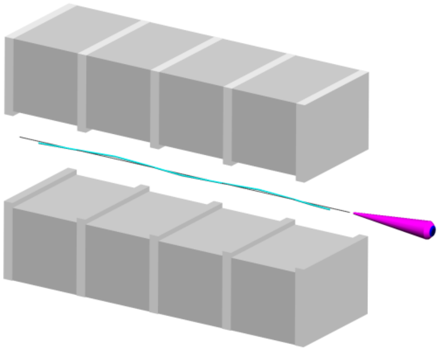
Ion channel protein



2003 Nobel Prize in Chemistry:

Roderick MacKinnon (Rockefeller Univ.)
1st K^+ channel structure by x-ray crystallography based on CHESS data (1998)

X-ray characteristics needed



• for properly tuned undulator: X-ray phase space is a replica from electron bunch + convolution with the diffraction limit

• ideally, one wants the phase space to be diffraction limited (i.e. full transverse coherence), e.g. $\varepsilon_{\perp,rms} = \lambda/4\pi$, or 0.1 Å for 8 keV X-rays (Cu K_α), or **0.1 μm** normalized at 5 GeV

Flux	ph/s/0.1%bw
Brightness	ph/s/mrad^2/0.1%bw
Brilliance	**ph/s/mm^2/mrad^2/0.1%bw**

Introduction

Let's review why ERL is a good idea for a light source

Critical electron beam parameters for X-ray production:

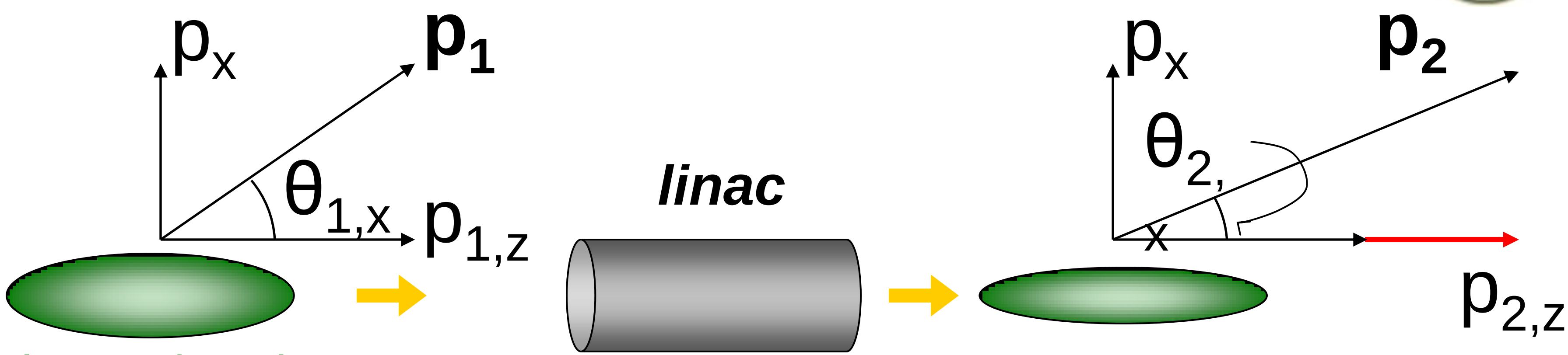
6D Phase Space Area:

- Horizontal Emittance {x, x'}
- Vertical Emittance {y, y'}
- Energy Spread & Bunch length {ΔE, t}

Number of Electrons / Bunch,

Bunch Rep Rate: I_{peak}, $I_{average}$

Introduction: adiabatic damping



electron bunch

$\boldsymbol{\varepsilon_1}$

$$\boldsymbol{\varepsilon_2} = \boldsymbol{\varepsilon_1}\frac{p_{1,z}}{p_{2,z}}$$

geometric $\{x, \theta_x\}$

$$\boxed{\boldsymbol{\varepsilon} = \frac{\boldsymbol{\varepsilon_n}}{\boldsymbol{\beta\gamma}}}$$

normalized $\{x, \frac{p_x}{mc^2}\}$

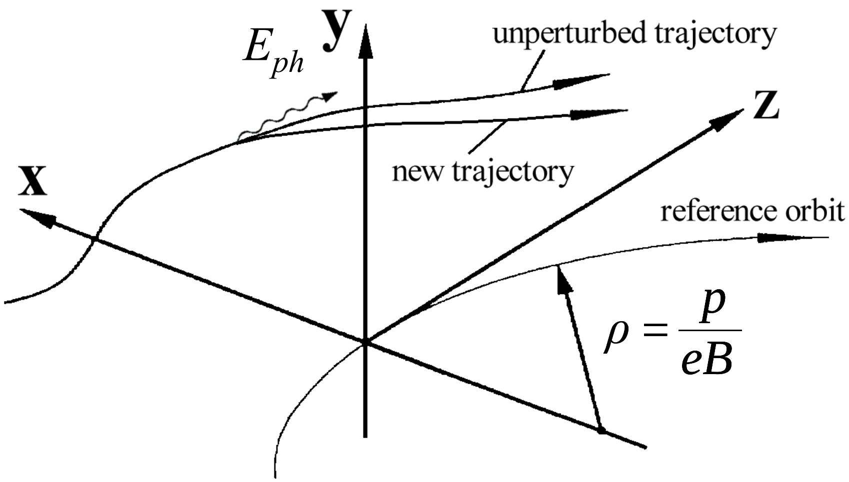
$\boldsymbol{\varepsilon_n}$ is invariant since

$\{x;\ p_x = mc^2\beta\gamma\cdot\theta_x\}$ form canonically conjugate variables

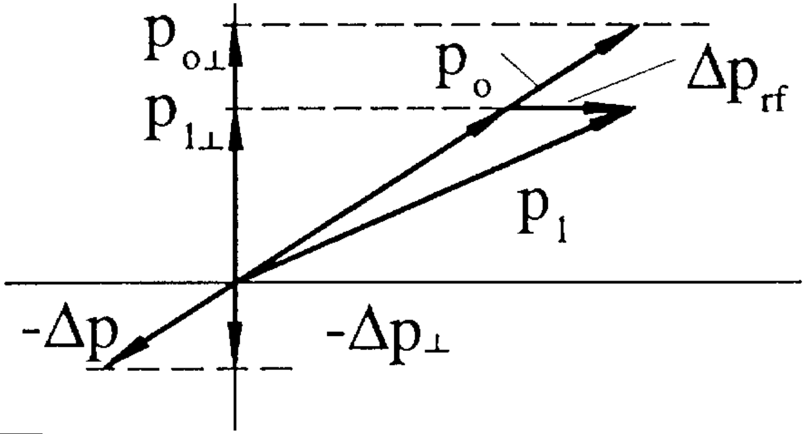
Introduction: storage rings (I)

Equilibrium

Quantum Excitation **vs.** **Radiative Damping**



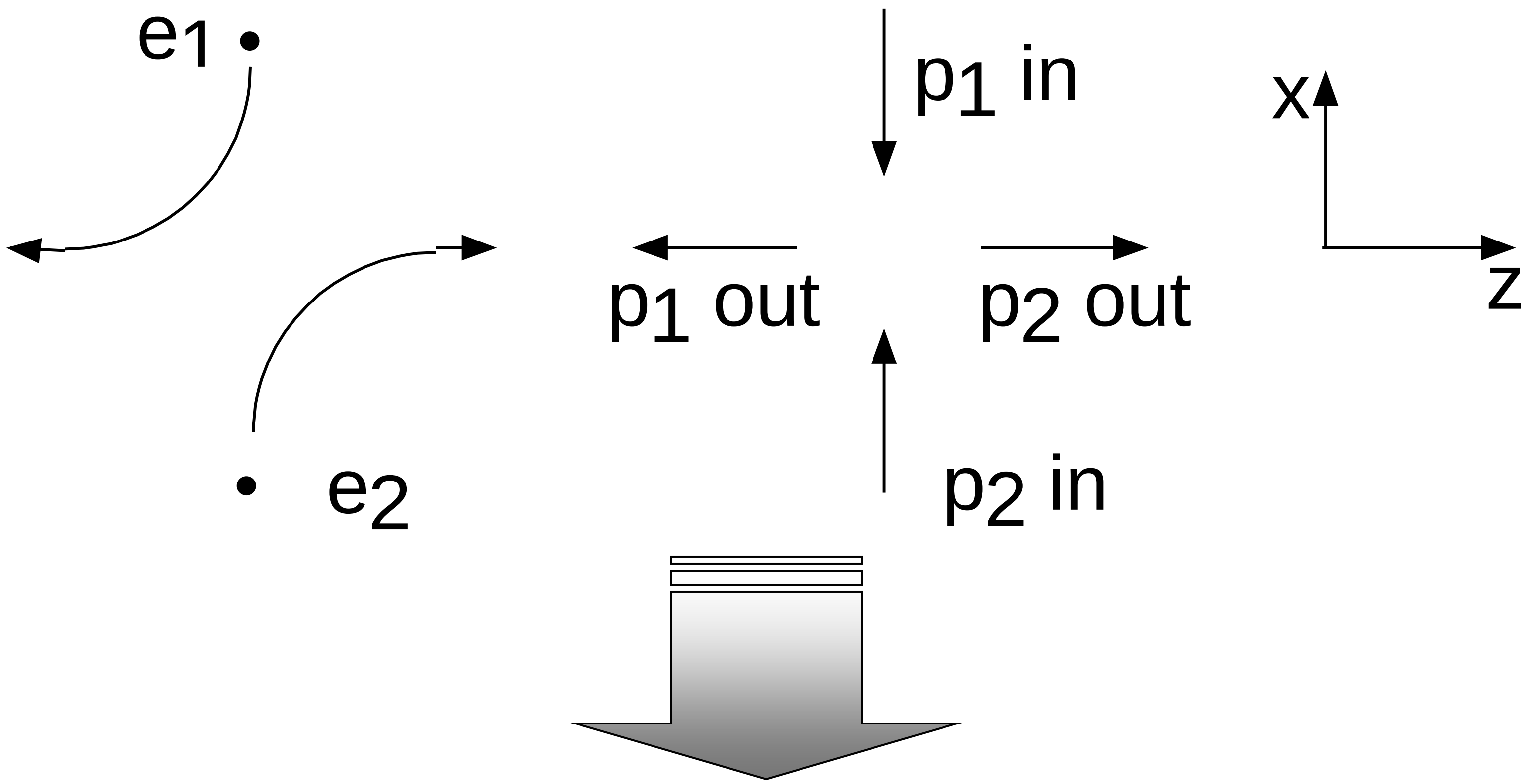
$$\frac{d\sigma_E^2}{dt} \sim \dot{N}_{ph} E_{ph}^2$$



Emittance (hor.), Energy Spread, Bunch Length

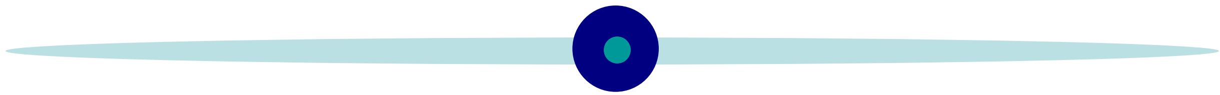
Introduction: storage rings (II)

Touschek Effect



Beam Lifetime vs. Space Charge Density

Why an ERL?

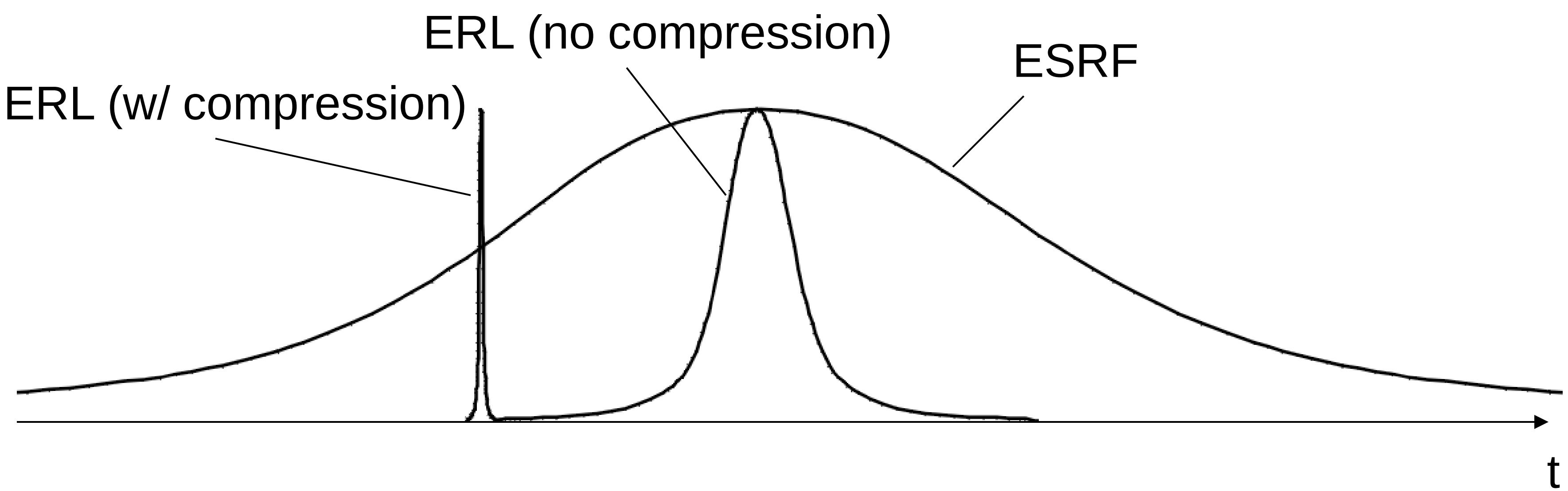


ESRF 6 GeV @ 200 mA

ε_x = 4 nm mrad
ε_y = 0.02 nm mrad
B ~ 10^{20} ph/s/mm^2/mrad^2/0.1%BW
L_{ID} = 5 m

ERL 5 GeV @ 10-100 mA

$\varepsilon_x = \varepsilon_y \rightarrow$ 0.01 nm mrad
B ~ 10^{23} ph/s/mm^2/mrad^2/0.1%BW
L_{ID} = 25 m



Comparing present and future sources

electron beam brilliance

$$I/\sqrt{\varepsilon_x^2+(\lambda/4\pi)^2}\sqrt{\varepsilon_y^2+(\lambda/4\pi)^2}$$

electron beam monochromaticity

$$1/5(\sigma_E/E)$$

A/(nm-rad)^2 × **max N_{und}**

A/(nm-rad)^2 compares brilliance from two short identical (K, N_{und}) undulators

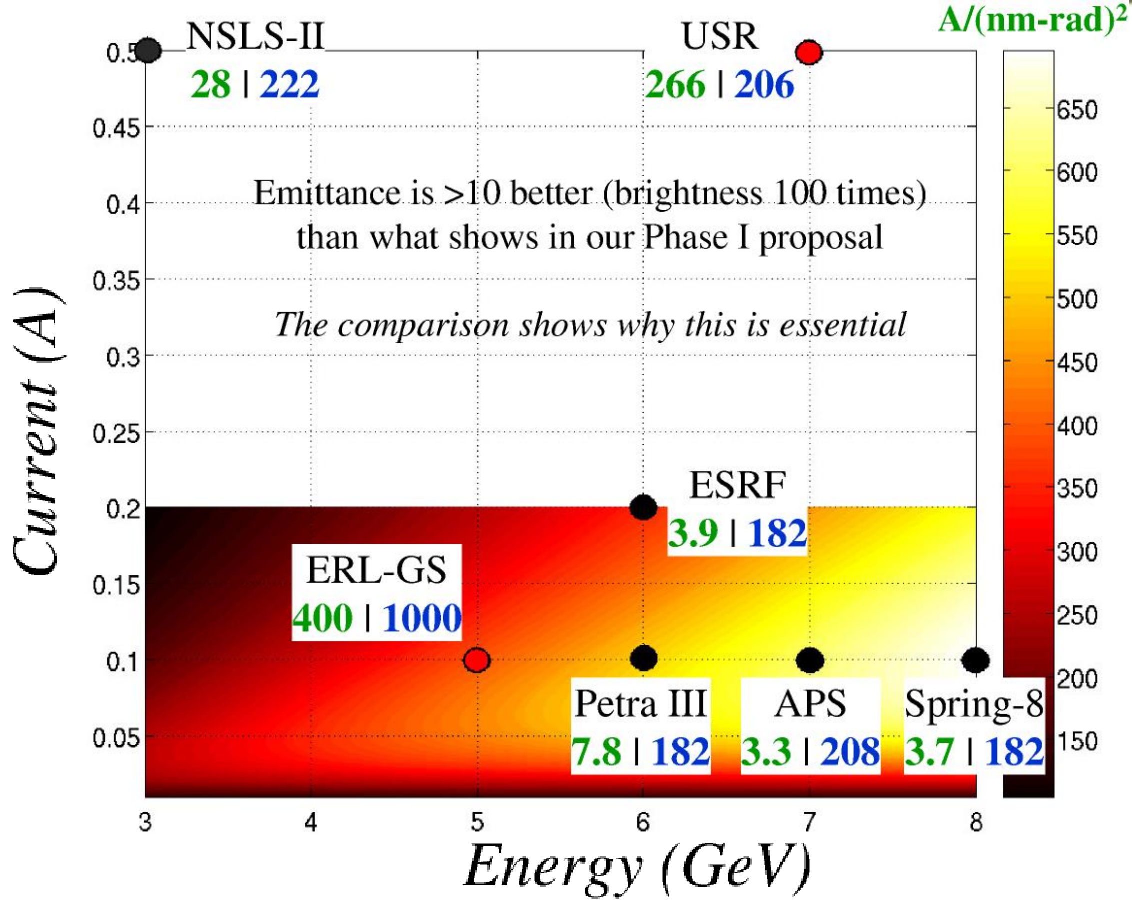
A/(nm-rad)^2 × **max N_{und}** compares maximum achievable brilliance

1 Angstrom brilliance comparison

ERL better by

Short IDs	
100	x ESRF
50	x PETRA
14	x NSLS-II
1.5	x USR

Max Length IDs	
560	x ESRF
280	x PETRA
64	x NSLS-II
7	x USR



ERL emittance is taken to be (PRSTAB **8** (2005) 034202)

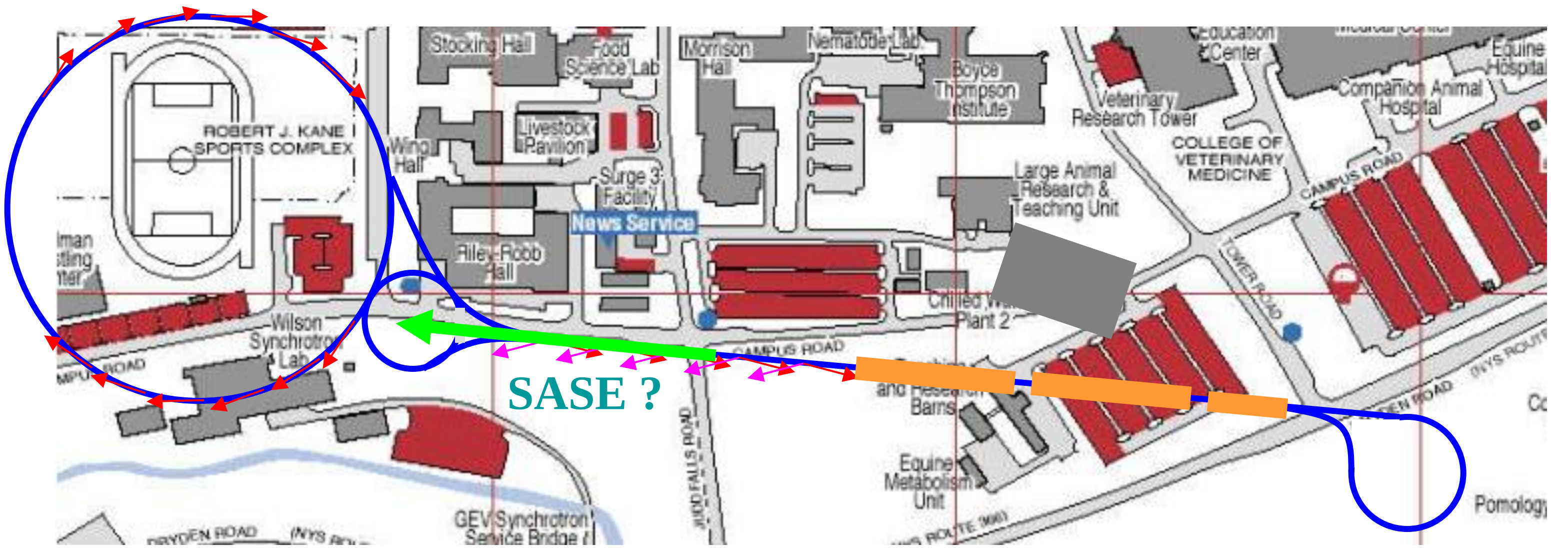
$$\varepsilon_n[\text{mm-mrad}] \approx (0.73+0.15/\sigma_z[\text{mm}]^{2.3}) \times q[\text{nC}]$$

plus a factor of 2 emittance growth for horizontal

Cornell vision of ERL light source

To continue the long-standing tradition of pioneering research in synchrotron radiation, Cornell University is carefully looking into constructing a first ERL hard x-ray light source.

But first…



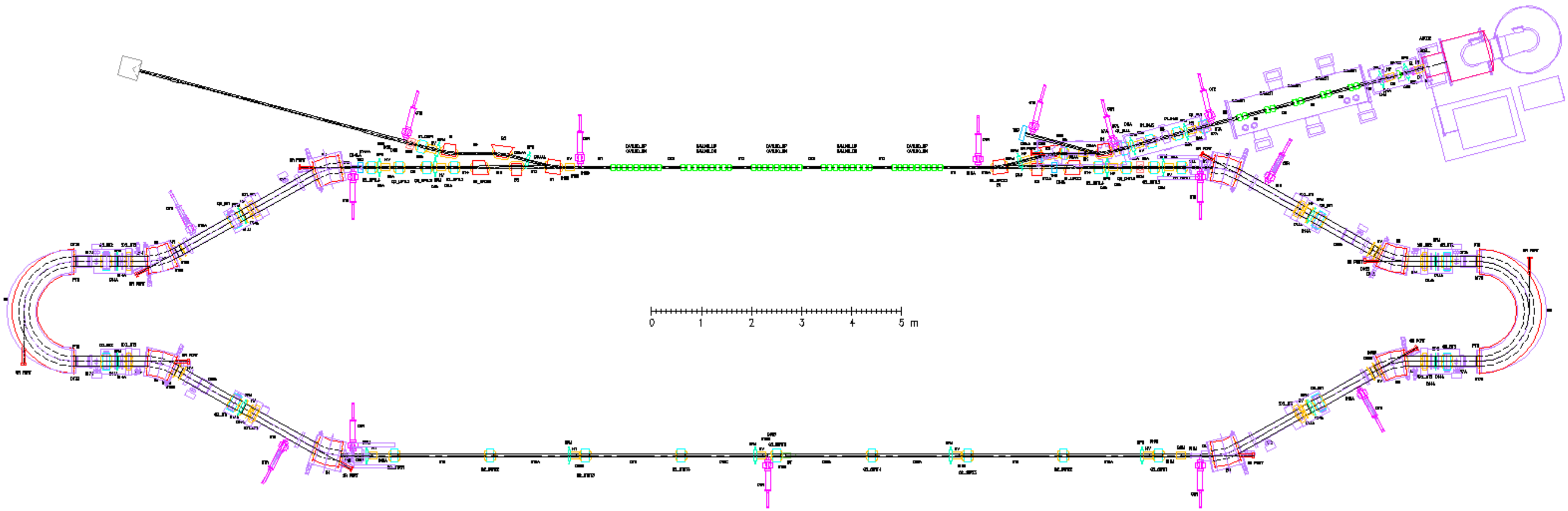
Need for the ERL prototype

Issues include:

- CW injector: produce $i_{avg} \geq 100$ mA, $q_{bunch} \sim 80$ pC @ 1300 MHz, $\varepsilon_n < 1$ mm mr, low halo with very good photo-cathode longevity.
- Maintain high Q and E_{acc} in high current beam conditions.
- Extract HOM's with very high efficiency ($P_{HOM} \sim 10$x previous).
- Control BBU by improved HOM damping, parameterize $i_{thr.}$
- How to operate with hi Q_L (control microphonics & Lorentz detuning).
- Produce + meas. $\sigma_t \sim 100$ fs with $q_{bunch} \sim 0.3$–0.4 nC ($i_{avg} < 100$ mA), understand / control CSR, understand limits on simultaneous brilliance and short pulses.
- Check, improve beam codes. Investigate multipass schemes.

Our conclusion: An ERL Prototype is needed to resolve outstanding technology and accelerator physics issues before a large ERL is built

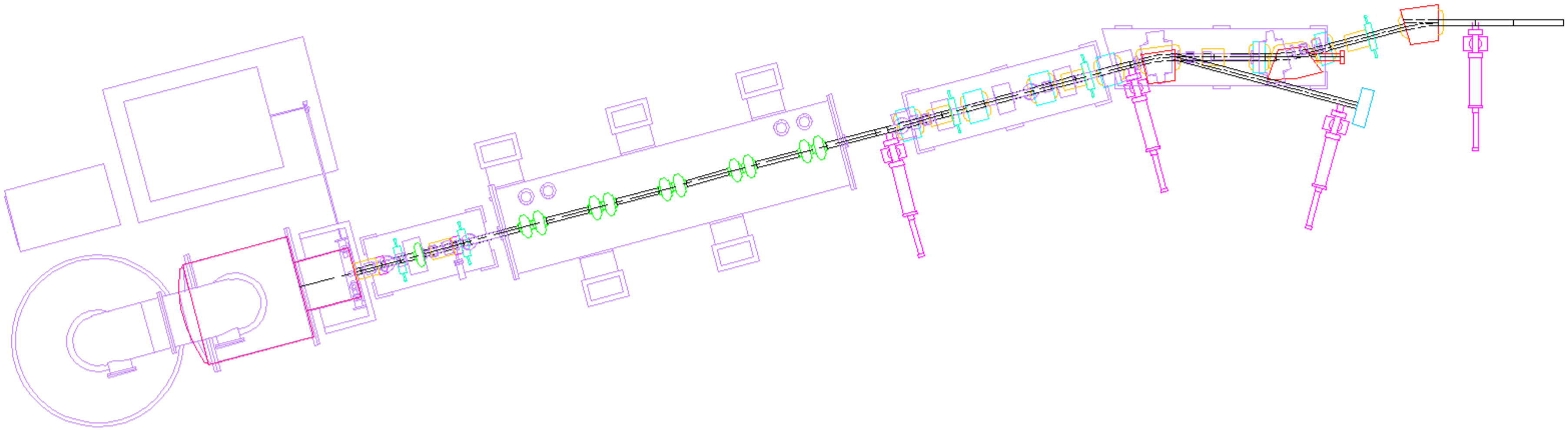
Cornell ERL Prototype



ERL PROTOTYPE

Energy	100 MeV	Injection Energy	5 – 15 MeV
Max Avg. Current	100 mA	E_{acc} @ Q_0	20 MeV/m @ 10^{10}
Charge / bunch	1 – 400 pC	Bunch Length	2 – 0.1 ps
Emittance (norm.)	$\leq$ 2 mm mr@77 pC		

Cornell ERL Phase I: Injector



Injector Parameters:

Beam Energy Range	5 – 15[a] MeV
Max Average Beam Current	100 mA
Max Bunch Rep. Rate @ 77 pC	1.3 GHz
Transverse Emittance, rms (norm.)	< 1[b] μm
Bunch Length, rms	2.1 ps
Energy Spread, rms	0.2 %

[a] at reduced average current
[b] corresponds to 77 pC/bunch

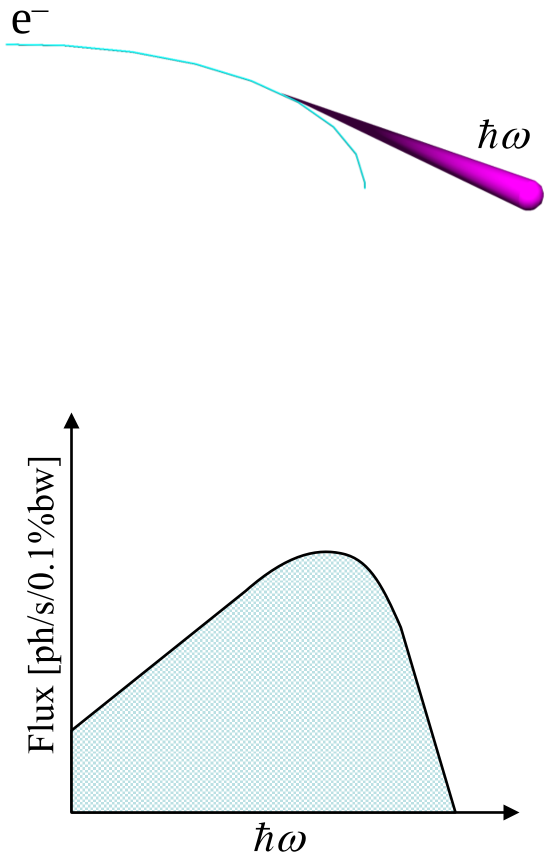
To learn more about Cornell ERL

Two web-sites are available

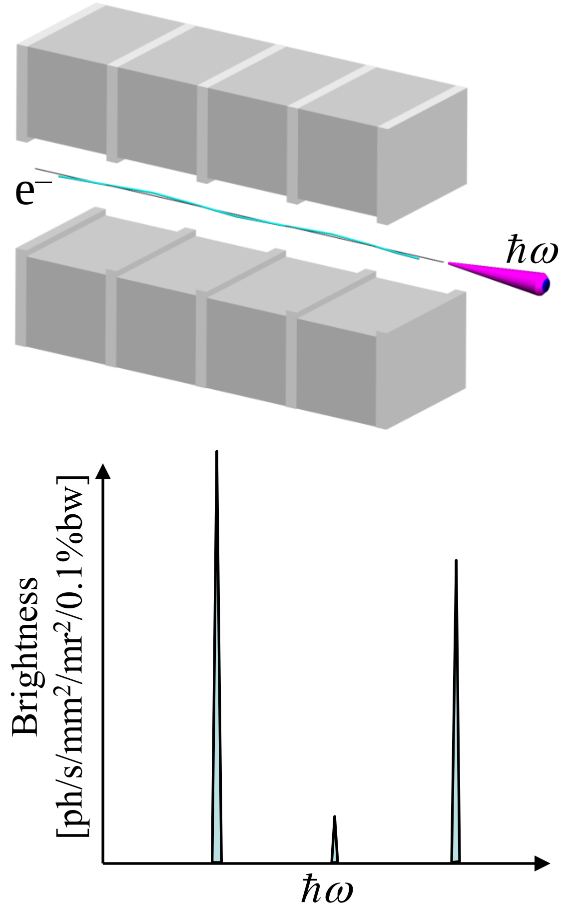
1) Information about Cornell ERL, X-ray science applications, other related projects worldwide
   http://erl.chess.cornell.edu/

2) ERL technical memorandum series
   http://www.lepp.cornell.edu/public/ERL/

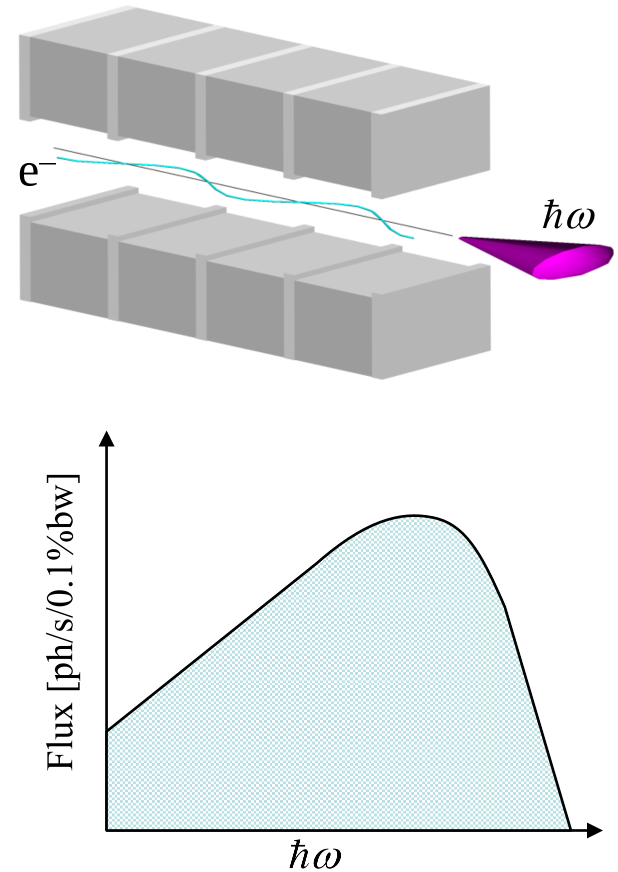
Bend Undulator Wiggler



white source

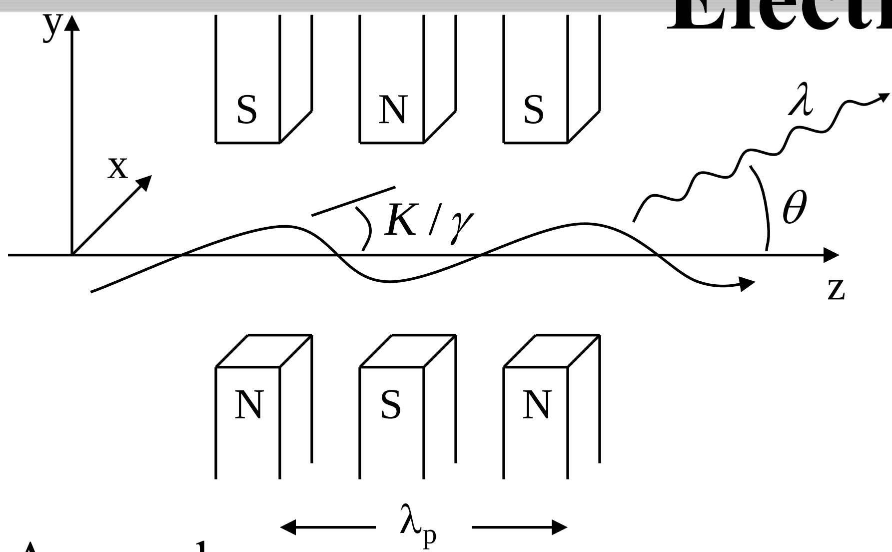


partially coherent source



powerful white source

Undulator Radiation from Single Electron



$$B_y = B_0 \sin k_p z$$

$$K = 93.4 B_0[\mathrm{T}]\lambda_p[\mathrm{m}]$$

Halbach permanent magnet undulator:

$$B_0[\mathrm{T}] \approx 3.33 \exp[-\kappa(5.47 - 1.8\kappa)]$$

for $SmCo_5$, here $\kappa = gap/\lambda_p$

Approaches:

1. Solve equation of motion (trivial), grab Jackson and calculate retarded potentials (not so trivial – usually done in the far field approximation). Fourier Transform the field seen by the observer to get the spectrum.

More intuitively in the electron rest frame:

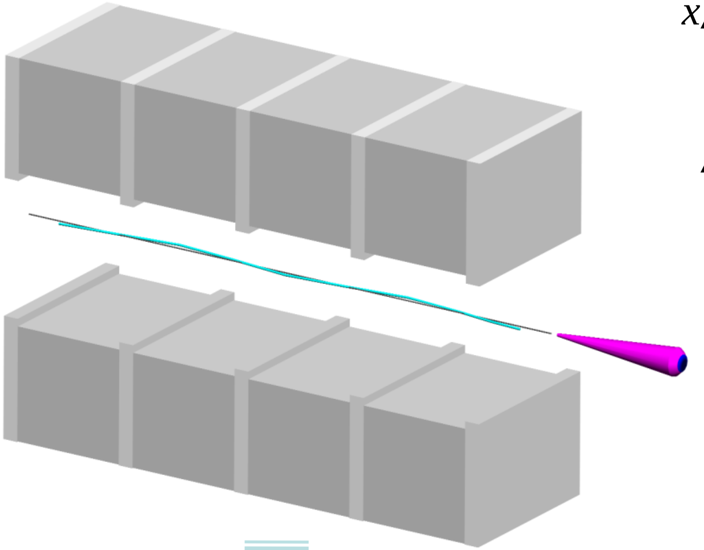
2. Doppler shift to the lab frame (nearly) simple harmonic oscillator radiation.

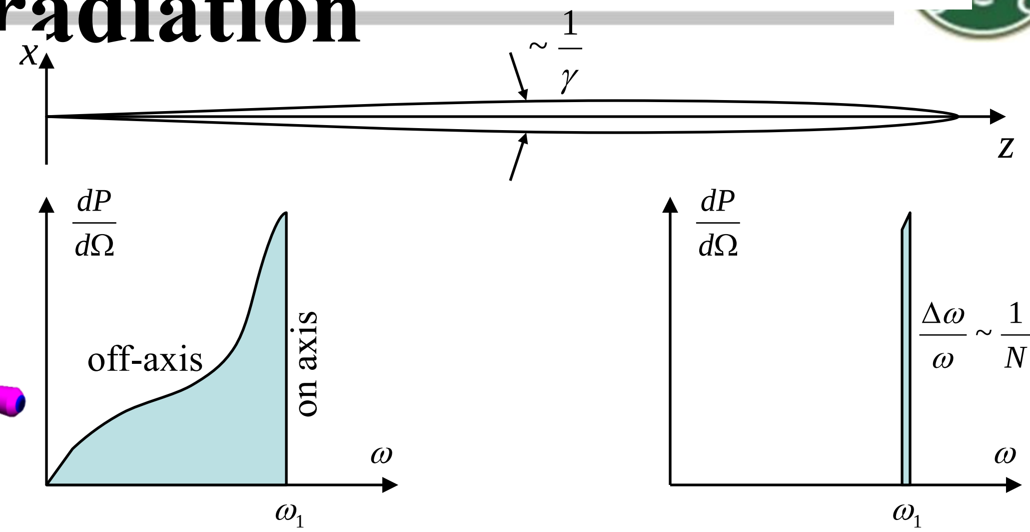
3. Doppler shift Thomson back-scattered undulator field "photons".

Or simply

4. Write interference condition of wavefront emitted by the electron.

Intuitive understanding of undulator radiation

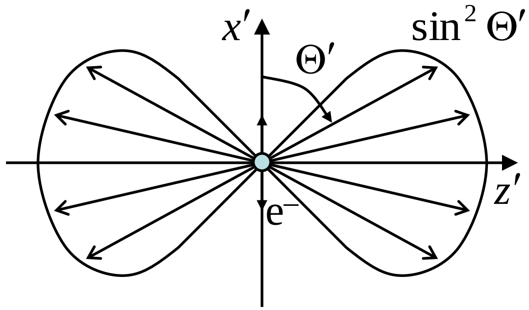




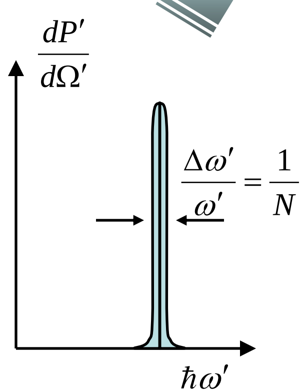
back to lab frame

after pin-hole aperture

$\lambda_p' = \lambda_p / \bar{\gamma}$



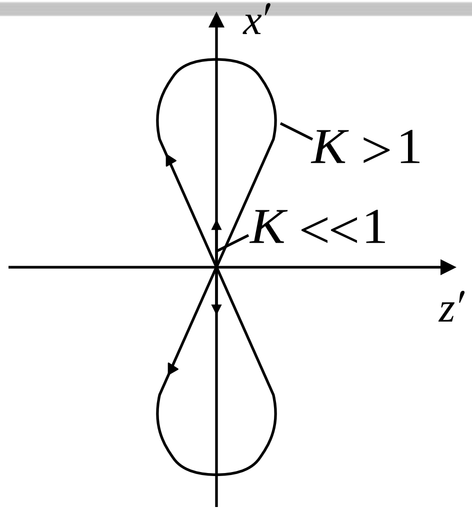
in e⁻ frame



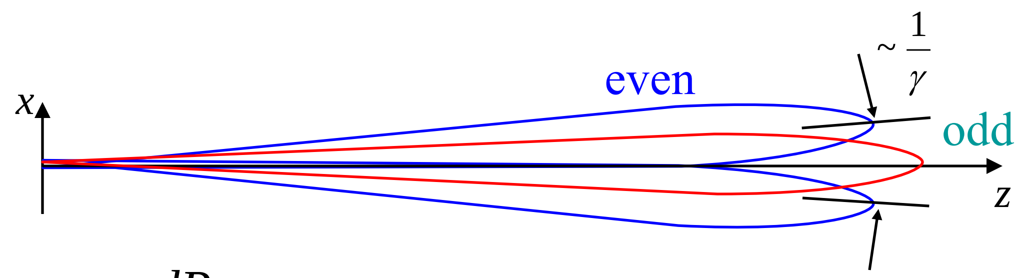
$$\boxed{\lambda_n = \frac{\lambda_p}{2\gamma^2 n}\left(1 + \tfrac{1}{2}K^2 + \gamma^2\theta^2\right)}$$

$$\boxed{\frac{\Delta\lambda}{\lambda_n} \sim \frac{1}{nN_p}}$$ (for fixed θ only!)

Higher Harmonics / Wiggler



motion in e$^-$ frame

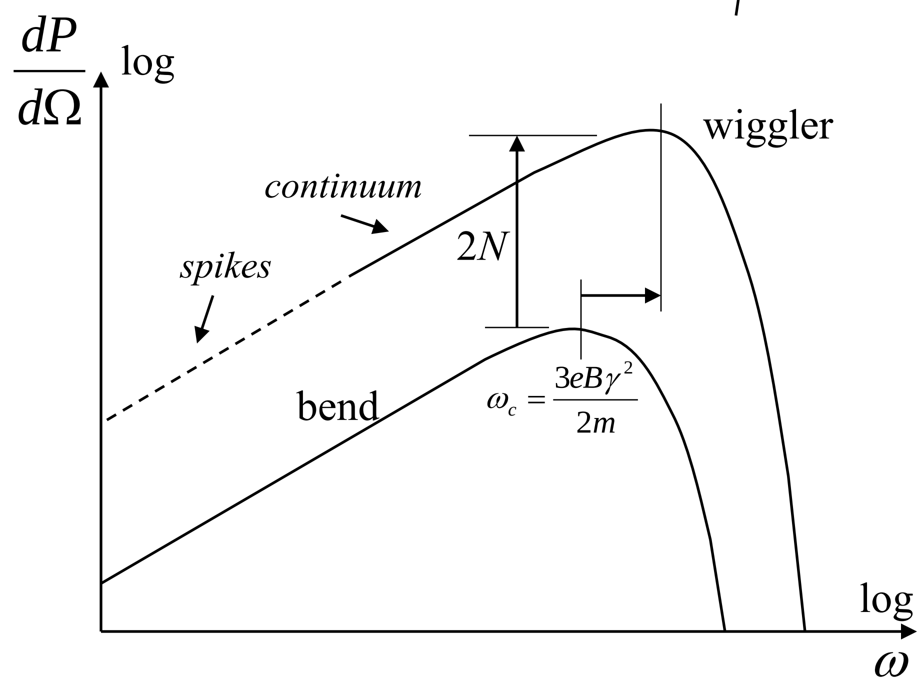


$K \leq 1$ undulator
$K > 1$ wiggler

$$n_c = \frac{3K}{4}\left(1 + \frac{K^2}{2}\right)$$

K	n_c
1	1
2	4
4	27
8	198
16	1548

critical harmonic number for wiggler
(in analogy to ω_c of bending magnet)



wiggler and bend spectra after pin-hole aperture

Total Radiation Power

$$\boxed{P_{tot} = \frac{\pi}{3}\alpha\hbar\omega_1 K^2(1+\tfrac{1}{2}K^2)N\frac{I}{e}}$$ or $$P_{tot}[\mathrm{W}] = 726\frac{E[\mathrm{GeV}]^2 K^2}{\lambda_p[\mathrm{cm}]^2}L[\mathrm{m}]I[\mathrm{A}]$$

e.g. about 1 photon from each electron in a 100-pole undulator, or 1 kW c.w. power from 1 m insertion device for beam current of 100 mA @ 5 GeV, $K = 1.5$, $\lambda_p = 2$ cm

---

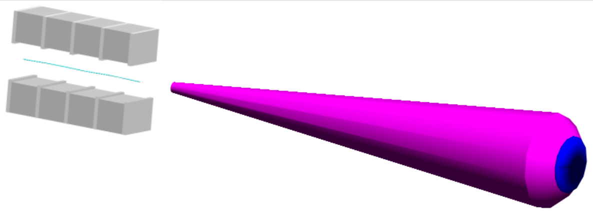
Note: the radiated power is independent from electron beam energy **if** one can keep $B_0\lambda_p \cong$ const, while $\lambda_p \sim \gamma^2$ to provide the same radiation wavelength. (e.g. low energy synchrotron and Thomson scattering light sources)

However, most of this power is discarded (bw ~ 1). Only a small fraction is used.

Radiation Needed

wavelength	0.1 – 2 Å (if a hard x-ray source)	
bw	$10^{-2} - 10^{-4}$	← *temporal coherence*
small source size & divergence		← *spatial coherence*

Undulator Central Cone



Select with a pin-hole aperture the cone:

$$\theta_{cen} = \frac{1}{2\gamma}\sqrt{\frac{1+\frac{1}{2}K^2}{nN}} = \sqrt{\frac{\lambda_n}{2L}}$$

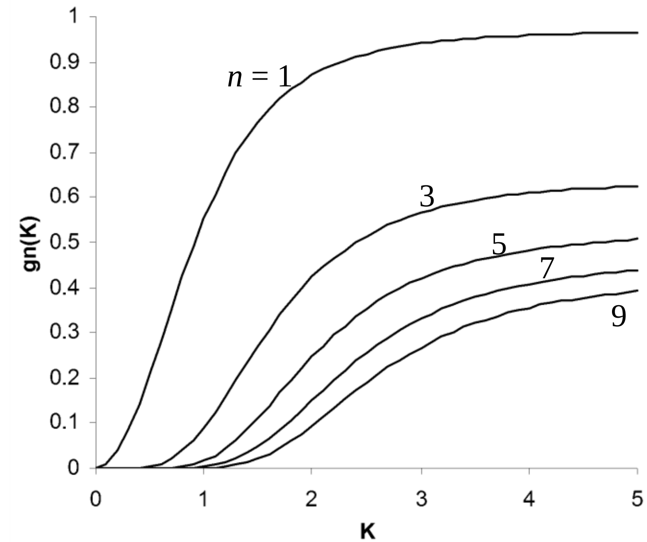
to get bw: $\dfrac{\Delta\omega}{\omega_n} \sim \dfrac{1}{nN}$

Flux in the central cone from n^{th} harmonic in bw $\Delta\omega/\omega_n$:

$$\dot{N}_{ph}\Big|_n = \pi\alpha N \frac{\Delta\omega}{\omega_n}\frac{I}{e} g_n(K) \le \boxed{\pi\alpha \frac{I}{e}\frac{g_n(K)}{n}}$$

Note: the number of photons in bw ~ $1/N$ is about 2 % max of the number of e⁻ for any-length undulator.

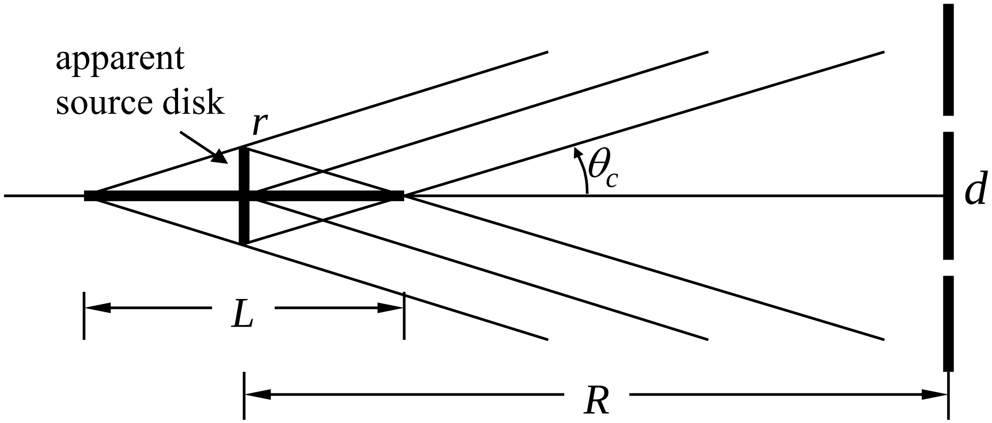
Undulator "efficiency": $\dfrac{P_{cen}}{P_{tot}} \le \dfrac{3g_n(K)}{K^2(1+\frac{1}{2}K^2)}\dfrac{1}{N_{\mathrm{p}}}$



Function $g_n(K) = \dfrac{nK^2[JJ]}{(1+\frac{1}{2}K^2)}$

A Word on Coherence of Undulator

Radiation contained in the central cone is transversely coherent (no beam emittance!)



Young's double-slit interference condition:

$$\frac{rd}{R} \sim \lambda$$

in Fraunhofer limit:

$$r \sim \theta_c L$$
$$\theta_c \sim r/R \quad \Rightarrow \theta_c \sim \sqrt{\lambda / L}$$

same as central cone

Spatial coherence (rms): $r \cdot \theta_c = \lambda / 4\pi$

Temporal coherence: $l_c = \lambda^2 / (2\Delta\lambda), \quad t_c = l_c / c$

Photon degeneracy: $\Delta_c = \dot{N}_{ph,c} t_c$

x-ray source	Δ_c
Rings	<1
ERLs	>1
XFEL	>>1

Next, we will study the effect of finite beam 6D emittance on undulator radiation.

Brightness Definition: Geometric Optics

Brightness is a measure of spatial (transverse) coherence of radiation. Spectral brightness (per 0.1 % BW) is usually quoted as a figure of merit, which also reflects temporal coherence of the beam. The word "spectral" is often omitted. Peak spectral brightness is proportional to photon degeneracy.

For the most parts we will follow K-J Kim's arguments regarding brightness definitions.

A ray coordinate in 4D phase space is defined as $\vec{x} = (x, y), \vec{\varphi} = (\varphi, \psi)$

$$B(\vec{x}, \vec{\varphi}; z) = \frac{d^4 F}{d^2\vec{x} d^2\vec{\varphi}}$$

Brightness is invariant in lossless linear optics as well as flux: $F = \int B(\vec{x}, \vec{\varphi}; z) d^2\vec{x} d^2\vec{\varphi}$

while flux densities are not: $\dfrac{d^2F}{d^2\vec{\varphi}} = \int B(\vec{x}, \vec{\varphi}; z) d^2\vec{x},\ \dfrac{d^2F}{d^2\vec{x}} = \int B(\vec{x}, \vec{\varphi}; z) d^2\vec{\varphi} \quad \neq inv$

Brightness Definition: Wave Optics

$$B(\vec{x},\vec{\varphi};z) = \frac{d\omega}{\hbar\omega}\frac{2\varepsilon_0 c}{T}\int d^2\vec{\xi}\left\langle E^*_{\omega,\varphi}(\vec{\varphi}+\vec{\xi}/2;z)E_{\omega,\varphi}(\vec{\varphi}-\vec{\xi}/2;z)\right\rangle e^{-ik\vec{\xi}\cdot\vec{x}}$$

$$= \frac{d\omega}{\hbar\omega}\frac{2\varepsilon_0 c}{\lambda^2 T}\int d^2\vec{y}\left\langle E^*_{\omega,x}(\vec{x}+\vec{y}/2;z)E_{\omega,x}(\vec{x}-\vec{y}/2;z)\right\rangle e^{-ik\vec{\varphi}\cdot\vec{y}}$$

here electric field in frequency domain is given in either coordinate or angular representation. Far-field (angular) pattern is equivalent to the Fourier transform of the near-field (coordinate) pattern:

$$E_{\omega,\varphi} = \frac{1}{\lambda^2}\int E_{\omega,x}(\vec{x};z)e^{-ik\vec{\varphi}\cdot\vec{x}}d^2\vec{x} \quad \Leftrightarrow \quad E_{\omega,x} = \int E_{\omega,\varphi}(\vec{x};z)e^{-ik\vec{\varphi}\cdot\vec{x}}d^2\vec{\varphi}$$

A word of caution: brightness as defined in wave optics may have negative values when diffraction becomes important. One way to deal with that is to evaluate brightness when diffraction is not important (e.g. z = 0) and use optics transform thereafter.

Diffraction Limit

Gaussian laser beam equation:

$$E(\vec{x},z)=E_0\frac{w_0}{w(z)}\exp\left\{i\left[kz-\cot\left(\frac{z}{z_R}\right)\right]-\vec{x}^2\left[\frac{1}{w^2(z)}-\frac{ik}{2R(z)}\right]\right\}$$

$$w^2(z)=w_0^2(1+z^2/z_R^2)$$

$$z_R=\pi w_0^2/\lambda$$

$$R(z)=z(1+z_R^2/z^2)$$

With corresponding brightness:

$$B(\vec{x},\vec{\varphi};z)=B_0\exp\left\{-\frac{1}{2}\left[\frac{(\vec{x}-z\vec{\varphi})^2}{\sigma_r^2}+\frac{\vec{\varphi}^2}{\sigma_{r'}^2}\right]\right\}$$

$$\sigma_r=w_0/2,\ \sigma_{r'}=1/kw_0$$

$$\sigma_r\sigma_{r'}=\lambda/4\pi$$

$$\sigma_r/\sigma_{r'}=z_R$$

$$B_0=\frac{F}{(2\pi\sigma_r\sigma_{r'})^2}$$

$$F_{coh}=\frac{B_0}{(\lambda/2)^2}$$

Effect of Electron Distribution

Previous result from undulator treatment:

$$E_{\omega,\varphi}(\vec{\varphi};0) = \frac{e}{4\pi\varepsilon_0 c}\frac{\omega}{\lambda\sqrt{2\pi}}\int dt' \mathrm{e}^{i\omega t(t')}\vec{n}\times(\vec{n}\times\vec{\beta}(t')), \quad \text{here} \ \ \vec{n} = (\vec{\varphi}, 1-\vec{\varphi}^2/2)$$

The field in terms of reference electron trajectory for i$^{\text{th}}$-electron is given by:

$$E^{\mathrm{i}}_{\omega,\varphi}(\vec{\varphi};0) = E^{0}_{\omega,\varphi}(\vec{\varphi}-\vec{\varphi}^{\mathrm{i}}_e;0)\mathrm{e}^{\underbrace{i\omega(t-\vec{\varphi}\cdot\vec{x}^{\mathrm{i}}_e/c)}_{\text{phase of i}^{\text{th}}\text{-electron}}}$$

For brightness we need to evaluate the following ensemble average for all electrons:

$$\left\langle E^{*}_{\omega,\varphi}(\vec{\varphi}_1;0)E_{\omega,\varphi}(\vec{\varphi}_2;0)\right\rangle = \sum_{\mathrm{i}=1}^{N_e}\left\langle E^{\mathrm{i}*}_{\omega,\varphi}(\vec{\varphi}_1;0)E^{\mathrm{i}}_{\omega,\varphi}(\vec{\varphi}_2;0)\right\rangle \qquad \propto N_e$$

$$+\sum_{\mathrm{i}\neq\mathrm{j}}\left\langle E^{\mathrm{i}*}_{\omega,\varphi}(\vec{\varphi}_1;0)E^{\mathrm{j}}_{\omega,\varphi}(\vec{\varphi}_2;0)\right\rangle \qquad \propto N_e(N_e-1)\,\mathrm{e}^{-k^2\sigma_z^2}$$

2$^{\text{nd}}$ term is the “FEL” term. Typically $N_e \mathrm{e}^{-k^2\sigma_z^2} << 1$, so only the 1$^{\text{st}}$ term is important.

Effect of Electron Distribution

$$\left\langle E^*_{\omega,\varphi}(\vec{\varphi}_1;0)E_{\omega,\varphi}(\vec{\varphi}_2;0)\right\rangle \approx N_e\left\langle \mathrm{e}^{ik\vec{x}_e^{\mathrm{i}}\cdot(\vec{\varphi}_1-\vec{\varphi}_2)}E^{0*}_{\omega,\varphi}(\vec{\varphi}_1-\vec{\varphi}_e^{\mathrm{i}};0)E^{0}_{\omega,\varphi}(\vec{\varphi}_2-\vec{\varphi}_e^{\mathrm{i}};0)\right\rangle$$

$$B(\vec{x},\vec{\varphi};0) = N_e\left\langle B^0(\vec{x}-\vec{x}_e^{\mathrm{i}},\vec{\varphi}-\vec{\varphi}_e^{\mathrm{i}};0)\right\rangle$$

$$= N_e\int B^0(\vec{x}-\vec{x}_e,\vec{\varphi}-\vec{\varphi}_e;0)f(\vec{x}_e,\vec{\varphi}_e;0)d^2\vec{x}_e d^2\vec{\varphi}_e$$

electron distribution

Brightness due to single electron has been already introduced. Total brightness becomes a convolution of single electron brightness with electron distribution function.

Brightness on axis due to single electron:

flux in the central cone

$$B^0(0,0;0) = \frac{F^0}{(\lambda/2)^2}$$

Finite Beam Emittance Effect

Oftentimes brightness from a single electron is approximated by Gaussian:

$$B^0(\vec{x},\vec{\varphi};0) = \frac{F^0}{(\lambda/2)^2}\exp\left\{-\frac{1}{2}\left[\frac{\vec{x}^2}{\sigma_r^2}+\frac{\vec{\varphi}^2}{\sigma_{r'}^2}\right]\right\}$$

$$\sigma_r = \sqrt{2\lambda L}/4\pi, \quad \sigma_{r'} = \sqrt{\lambda/2L}$$

Including the electron beam effects, amplitude and sigma's of brightness become:

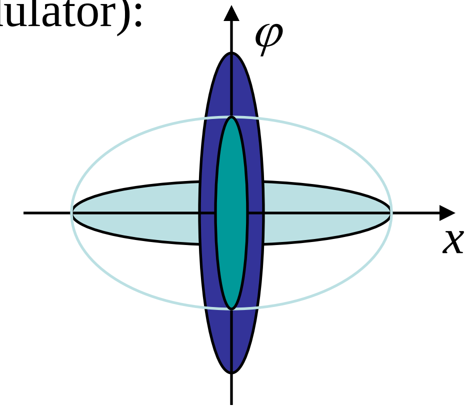
$$B(0,0;0) = \frac{F}{(2\pi)^2\sigma_{Tx}\sigma_{Tx'}\sigma_{Ty}\sigma_{Ty}}$$

$$\sigma_{Tx}^2 = \sigma_r^2 + \sigma_x^2 + a^2 + \tfrac{1}{12}\sigma_{x'}^2 L^2 + \tfrac{1}{36}\varphi^2 L^2 \qquad \sigma_{Tx'}^2 = \sigma_{r'}^2 + \sigma_{x'}^2$$

$$\sigma_{Ty}^2 = \sigma_r^2 + \sigma_y^2 + \tfrac{1}{12}\sigma_{y'}^2 L^2 + \tfrac{1}{36}\psi^2 L^2 \qquad \sigma_{Ty'}^2 = \sigma_{r'}^2 + \sigma_{y'}^2$$

Matching Electron Beam

Matched β-function is given by (beam waist at the center of undulator):



$$\beta_{x,y}^{opt} = \sigma_r / \sigma_{r'} = L / 2\pi$$

Brightness on axis becomes:

$$B(0,0;0) = \frac{F}{(\lambda/2)^2} \boxed{\frac{1}{\left(1 + \frac{\varepsilon_x}{\lambda/4\pi}\right)\left(1 + \frac{\varepsilon_y}{\lambda/4\pi}\right)}}$$

transversely coherent fraction of the central cone flux

Matched β-function has a broad minimum (for $\varepsilon/(\lambda/4\pi) << 1$ or $\varepsilon/(\lambda/4\pi) >> 1$)

$$\sigma_T \sigma_{T'} = \begin{cases} \sqrt{2}\min & \text{for} \quad \beta \approx 2L\varepsilon/\lambda \\ \min & \text{for} \quad \beta = L/2\pi \\ \sqrt{2}\min & \text{for} \quad \beta \approx \lambda L/(8\pi^2\varepsilon) \end{cases}$$

also if $\varepsilon \sim \lambda/4\pi \Rightarrow$

$\beta \approx 6\beta^{opt} \approx L$ is still acceptable

Energy Spread of the Beam

Energy spread of the beam can degrade brightness of undulators with many periods.

If the number of undulator periods is much greater than $N_\delta \approx 0.2/\sigma_\delta$, brightness will not grow with the number of periods.

Maximal spectral brightness on axis becomes

$$B(0,0;0) = \frac{F}{(\lambda/2)^2} \frac{1}{\left(1+\dfrac{\varepsilon_x}{\lambda/4\pi}\right)\left(1+\dfrac{\varepsilon_y}{\lambda/4\pi}\right)} \frac{1}{\sqrt{1+\left(\dfrac{N}{N_\delta}\right)^2}}$$

Photon Degeneracy

Number of photons in a single quantum mode:

$$\hbar k \sigma_x \sigma_\varphi \approx \frac{\hbar}{2} \qquad \hbar k \sigma_y \sigma_\psi \approx \frac{\hbar}{2} \qquad \sigma_E \sigma_t \approx \frac{\hbar}{2}$$

Peak brightness is a measure of photon degeneracy

$$\Delta_c = B_{peak}\left(\frac{\lambda}{2}\right)^3 \frac{\Delta\lambda}{\lambda}\frac{1}{c}$$

E.g. maximum photon degeneracy that is available from undulator (non-FEL)

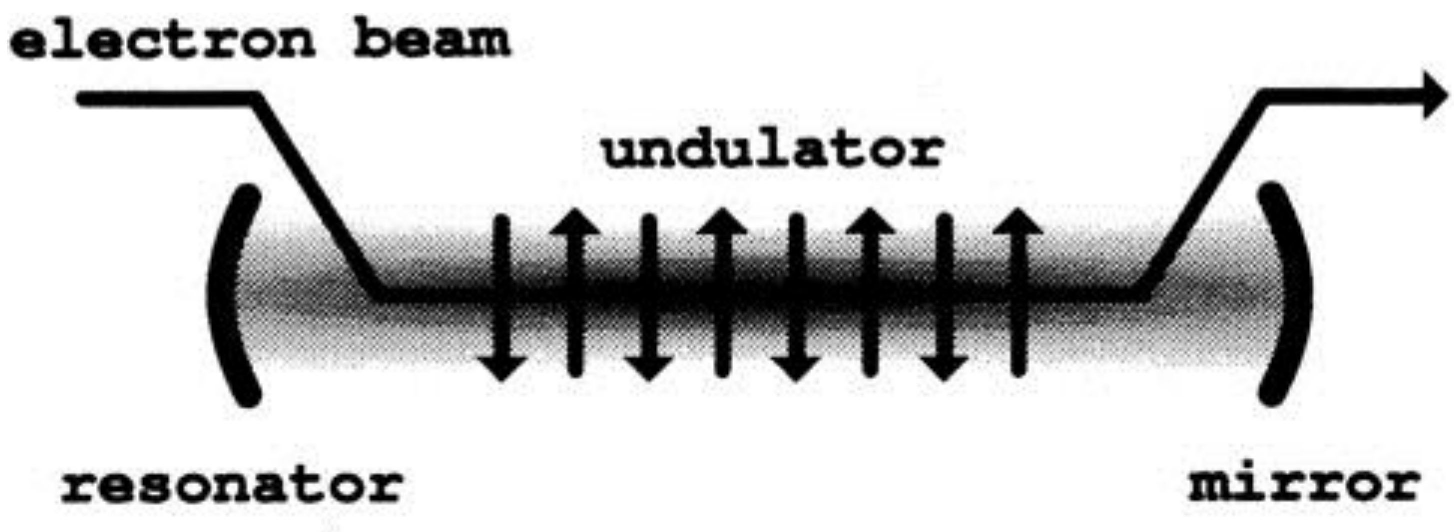
$$\Delta_c^{\max} \approx \alpha \frac{\lambda_n}{\sigma_z} N_e N \cdot g_n(K) \quad \text{more typically, however: } \Delta_c \approx 10^{-3} \alpha \frac{\lambda_n^3}{\varepsilon_x \varepsilon_y \varepsilon_z} N_e \frac{g_n(K)}{n}$$

diffraction-limited emittance dominated

More reading on synchrotron radiation

1. K.J. Kim, Characteristics of Synchrotron Radiation, AIP Conference Proceedings **189** (1989) pp.565-632
2. R.P. Walker, Insertion Devices: Undulators and Wigglers, CERN Accelerator School **98-04** (1998) pp.129-190, and references therein. Available on the Internet at http://preprints.cern.ch/cernrep/1998/98-04/98-04.html
3. B. Lengeler, Coherence in X-ray physics, Naturwissenschaften **88** (2001) pp. 249-260, and references therein.
4. D. Attwood, Soft X-rays and Extreme UV Radiation: Principles and Applications, Cambridge University Press, 1999. Chapters 5 (Synchrotron Radiation) and 8 (Coherence at Short Wavelength) and references therein.

Oscillator FEL



Free Electron Laser, Optical Klystron

- Principle
  - Stimulate emission of EM radiation from relativistic electron beam through interaction with an external EM field
  - Make electrons move against wave EM field to loose energy and amplify wave
  - How?

To the end of this lecture is taken from H. Wiedemann, USPAS, Jan 19-24, 2004, College of William and Mary slides 293-313 energy gain/loss of electron from/to EM field

$$\Delta W = -e\int \overrightarrow{E_{\mathrm{L}}}\,\mathrm{d}\vec{s} = -e\int \vec{v}\,\overrightarrow{E_{\mathrm{L}}}\,\mathrm{d}t = 0$$

because $\vec{v} \perp \overrightarrow{E_{\mathrm{L}}}$

how do we get better coupling ?

need particle motion in the direction of electric field from EM wave

undulator

Trajectory

$$\frac{d^2x}{dt^2} = -\frac{eB_0}{mc\gamma}\frac{dz}{dt}\cos k_p z$$
$$\frac{d^2z}{dt^2} = +\frac{eB_0}{mc\gamma}\frac{dx}{dt}\cos k_p z$$

$$\frac{dx}{dt} = -c\beta\frac{K}{\gamma}\sin k_p z$$
$$\frac{dz}{dt} = +c\beta\left(1-\frac{K^2}{2\gamma^2}\sin^2 k_p z\right)$$

drift velocity

$$\bar{\beta} = \beta\left(1-\frac{K^2}{4\gamma^2}\right)$$

$$x(t) = a\cos(k_p c\bar{\beta}t)$$

$$a = \frac{K}{\gamma k_p}$$

$$z(t) = c\bar{\beta}t + \frac{1}{8}k_p a^2 \sin(2k_p c\bar{\beta}t)$$

Energy Transfer

$$\Delta W = -e\int v_x E_{xL}\,\mathrm{d}t = -e\int\left[c\frac{K}{\gamma}\sin\left(k_{\mathrm{u}}s\right)\right]\left[E_{xL,0}\cos\left(k_{\mathrm{L}}s-\omega_{\mathrm{L}}t+\varphi_0\right)\right]\mathrm{d}t$$

$$= -\frac{ecKE_{xL,0}}{\gamma}\int\Big\{\sin\left[\left(k_{\mathrm{L}}+k_{\mathrm{u}}\right)s-\omega_{\mathrm{L}}t+\varphi_0\right]$$

$$\cancel{-\sin\left[\left(k_{\mathrm{L}}-k_{\mathrm{u}}\right)s-\omega_{\mathrm{L}}t+\varphi_0\right]}\Big\}\mathrm{d}t$$

get continuous energy transfer if $\Psi_{\pm} = \left(k_{\mathrm{L}} \pm k_{\mathrm{u}}\right)\bar{s} - \omega_{\mathrm{L}}t + \varphi_0 \approx \mathrm{const.}$

$$\frac{\mathrm{d}\Psi_{\pm}}{\mathrm{d}t} = \left(k_{\mathrm{L}}+k_{\mathrm{u}}\right)\frac{\mathrm{d}\bar{s}}{\mathrm{d}t} - \omega_{\mathrm{L}} \approx 0 = \left(k_{\mathrm{L}}+k_{\mathrm{u}}\right)\beta\left(1-\frac{K^2}{4\gamma^2}\right) - k_{\mathrm{L}}$$

condition for continuous energy transfer

$$k_{\mathrm{u}} = \frac{k_{\mathrm{L}}}{2\gamma^2}\left(1+\tfrac{1}{2}K^2\right) \quad \text{or} \quad \lambda_{\mathrm{L}} = \frac{\lambda_{\mathrm{u}}}{2\gamma^2}\left(1+\tfrac{1}{2}K^2\right)$$

Gain/loss per unit path length

$$\frac{\mathrm{d}\gamma}{\mathrm{d}s} = \frac{\mathrm{d}W}{c\mathrm{d}t}\frac{1}{mc^2} = -\frac{ecKE_{x\mathrm{L},0}}{2\gamma mc^2}\sin\left[\left(k_\mathrm{L}+k_\mathrm{u}\right)s - \omega_\mathrm{L} t + \varphi_0\right]$$

where $s = ct\bar{\beta} + \frac{K^2}{8\gamma^2 k_\mathrm{u}}\sin\left(2k_\mathrm{u}ct\right)$

define $\eta = \frac{k_\mathrm{L}K^2}{8\gamma^2 k_\mathrm{u}}$ and $K_\mathrm{L} = \frac{eE_{x\mathrm{L},0}}{k_\mathrm{u}mc^2}$

and the energy gain becomes

$$\frac{\mathrm{d}\gamma}{\mathrm{d}s} = -\frac{k_\mathrm{u}K_\mathrm{L}K}{2\gamma}\left[J_0(\eta) - J_1(\eta)\right]\sin\left[\left(k_\mathrm{L}+k_\mathrm{u}\right)\bar{s} - \omega_\mathrm{L} t + \varphi_0\right]$$

the phase varies slowly for particles off the resonance energy $\gamma_\mathrm{r}^2 = \frac{k_\mathrm{L}}{2k_\mathrm{u}}\left(1+\frac{1}{2}K^2\right)$

$$\frac{\mathrm{d}\Psi}{\mathrm{d}s} = k_\mathrm{u}\left(1-\frac{\gamma_\mathrm{r}^2}{\gamma^2}\right) = 2\frac{k_\mathrm{u}}{\gamma_\mathrm{r}}\Delta\gamma$$

where $\Delta\gamma = \gamma - \gamma_\mathrm{r}$

Pendulum Equation

$$\frac{d\Delta\gamma}{ds} = \frac{d\gamma}{ds} - \frac{d\gamma_r}{ds} = -\frac{k_u K_L K}{2\gamma}[J_0(\eta) - J_1(\eta)] \sin\Psi$$

$$\frac{d^2\Psi}{ds^2} = 2\frac{k_u}{\gamma_r}\frac{d\Delta\gamma}{ds} = -\frac{k_u^2 K_L K}{\gamma_r \gamma}[J_0(\eta) - J_1(\eta)] \sin\Psi$$

Pendulum equation $$\boxed{\frac{d^2\Psi}{ds^2} + \Omega_L^2 \sin\Psi = 0}$$

with $$\Omega_L^2 = \frac{k_u^2 K_L K}{\gamma_r \gamma}[J_0(\eta) - J_1(\eta)]$$

Gain

gain of laser field: $$\Delta W_{\mathrm{L}} = -mc^2 \Delta\gamma$$

stored energy in laser field $$W_{\mathrm{L}} = \frac{1}{2}\epsilon_0 E_{\mathrm{L},0}^2 V$$

gain of laser field per electron

$$G_1 = \frac{\Delta W_{\mathrm{L}}}{W_{\mathrm{L}}} = -\frac{2mc^2}{\epsilon_0 E_{\mathrm{L},0}^2 V}\Delta\gamma = -\frac{mc^2\gamma_{\mathrm{r}}}{\epsilon_0 E_{\mathrm{L},0}^2 V k_{\mathrm{u}}}\Delta\Psi'$$

or for all electrons $$G = -\frac{e^2 k_{\mathrm{u}} K^2}{\epsilon_0 mc^2}\frac{n_{\mathrm{b}}}{\gamma_{\mathrm{r}}^3}\left[J_0(\eta) - J_1(\eta)\right]^2 \frac{\langle\Delta\Psi'\rangle}{\Omega_{\mathrm{L}}^4}$$

where n_{b} is the electron density

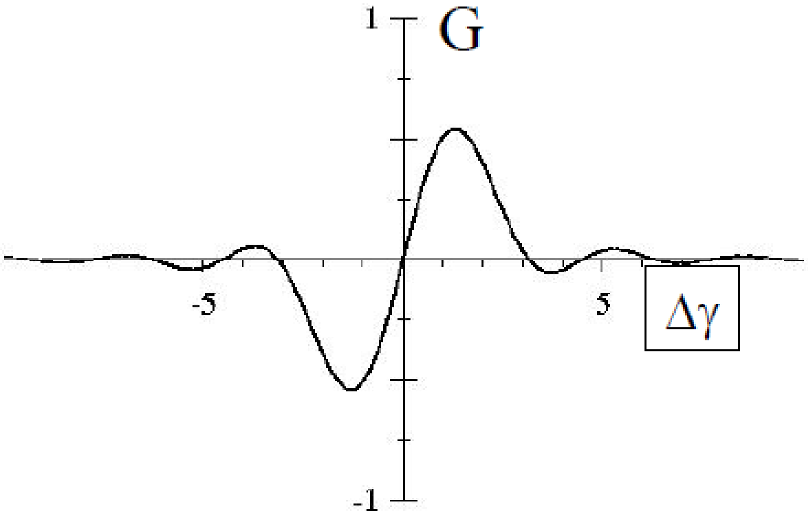
the average variation of $\langle\Delta\Psi'\rangle$ can be calculated from the phase equation

Gain Curve

FEL gain per pass $G = -\frac{\pi e^2 K^2 N_u \lambda_u^2}{4\epsilon_0 m c^2} \frac{n_b}{\gamma_r^3} [J_0(\eta) - J_1(\eta)]^2 \frac{d}{dw} (\frac{\sin w}{w})^2$

with $w = \frac{2\pi N_u}{\gamma_r} \left(\gamma_0 - \gamma_r\right)$ and $\eta = \frac{k_L K^2}{8\gamma^2 k_u}$

gain curve $\frac{d}{dw} (\frac{\sin w}{w})^2$



for finite gain: $\Delta\gamma \neq 0$

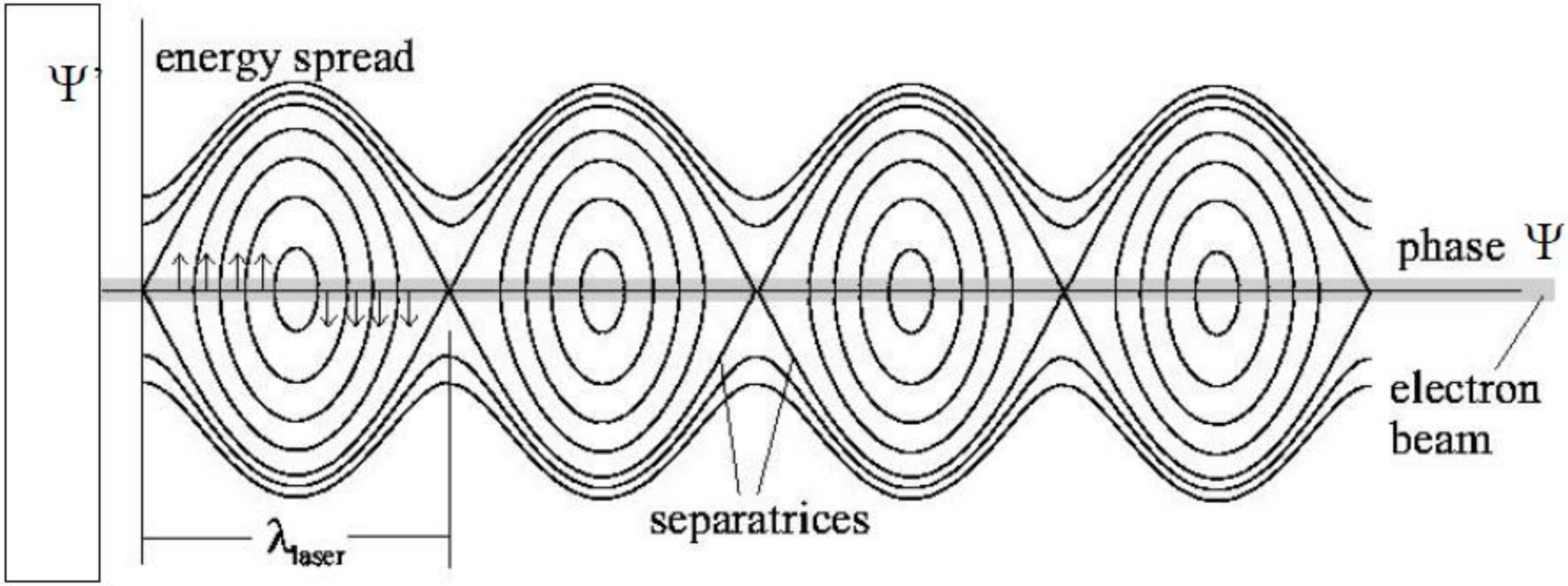
adjust beam energy slightly higher than resonance energy

laser energy $W_L = W_{L,0} e^{Gn}$ n number of passes

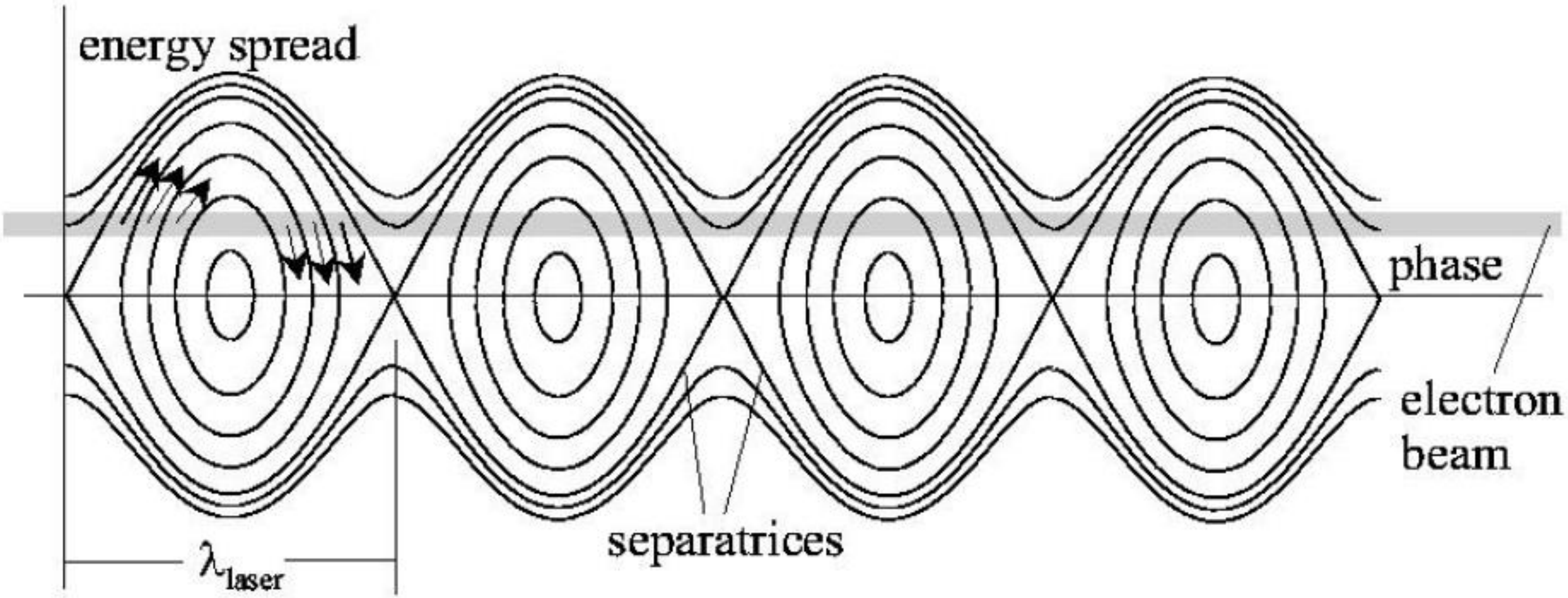
Phase Space Motion

Pendulum equation $\frac{d^2\Psi}{ds^2} + \Omega_L^2 \sin\Psi = 0$ | $\cdot\Psi'$ and integrate

$$\frac{1}{2}\Psi'^2 - \Omega_L^2 \cos\Psi = \text{const}$$

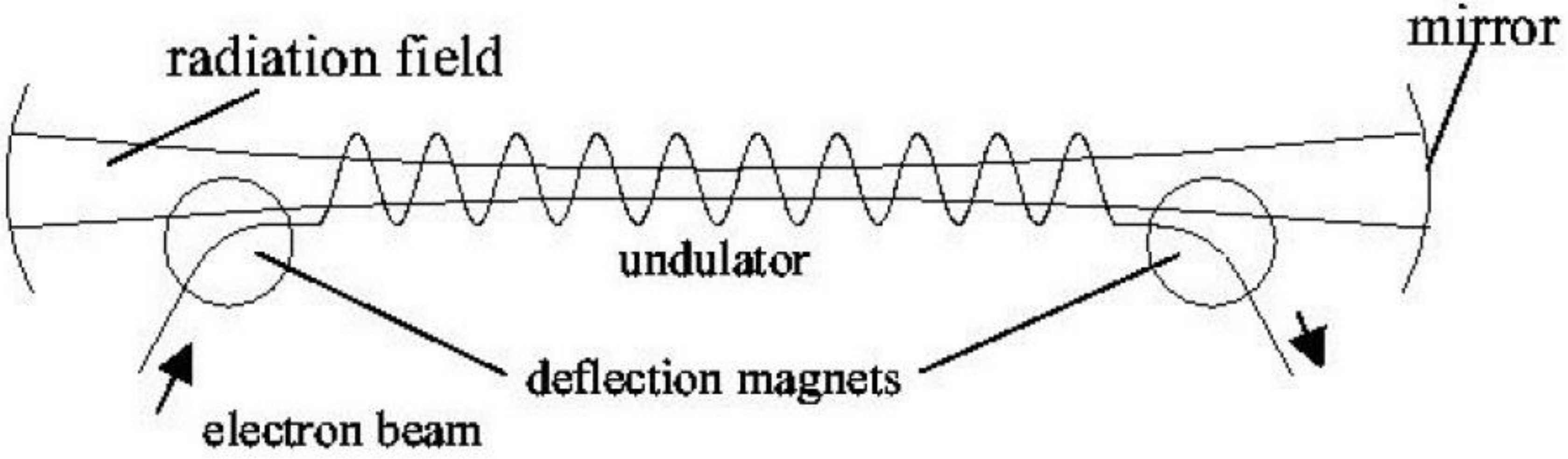


no net energy transfer !



for $\gamma_0 > \gamma_r$ energy transfer to laser field !

FEL Schematically



Electron Motion

velocity of wave: c

average drift velocity of electron: $\bar{\beta} = \beta\left(1 - \frac{K^2}{4\gamma^2}\right)$

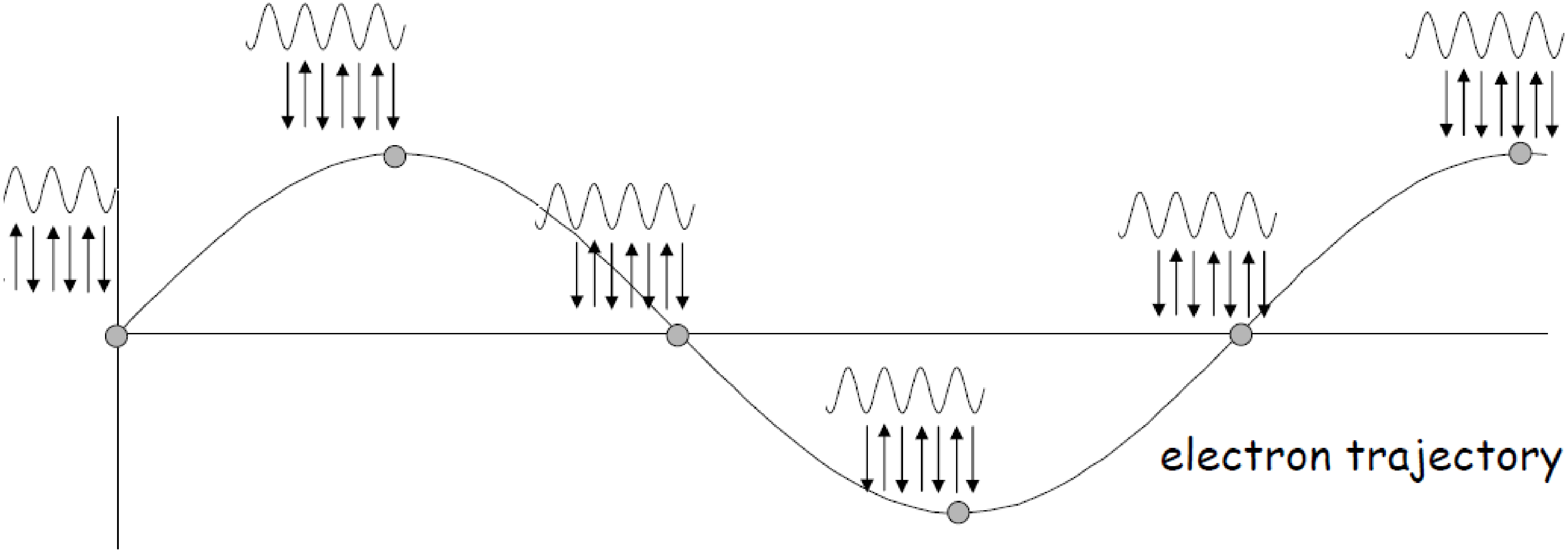
time for electron to travel one period: $\tau = \frac{\lambda_u}{c\bar{\beta}} = \frac{\lambda_u}{c\beta\left(1-\frac{K^2}{4\gamma^2}\right)}$

distance wave propagates in time τ : $s_\gamma = \frac{\lambda_u c}{c\beta\left(1-\frac{K^2}{4\gamma^2}\right)}$

$$\delta s = \frac{\lambda_u}{\beta\left(1-\frac{K^2}{4\gamma^2}\right)} - \lambda_u \approx \lambda_u\left[\frac{1}{\beta}\left(1 + \frac{K^2}{4\gamma^2}\right) - 1\right] \approx \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{1}{2}K^2\right)$$

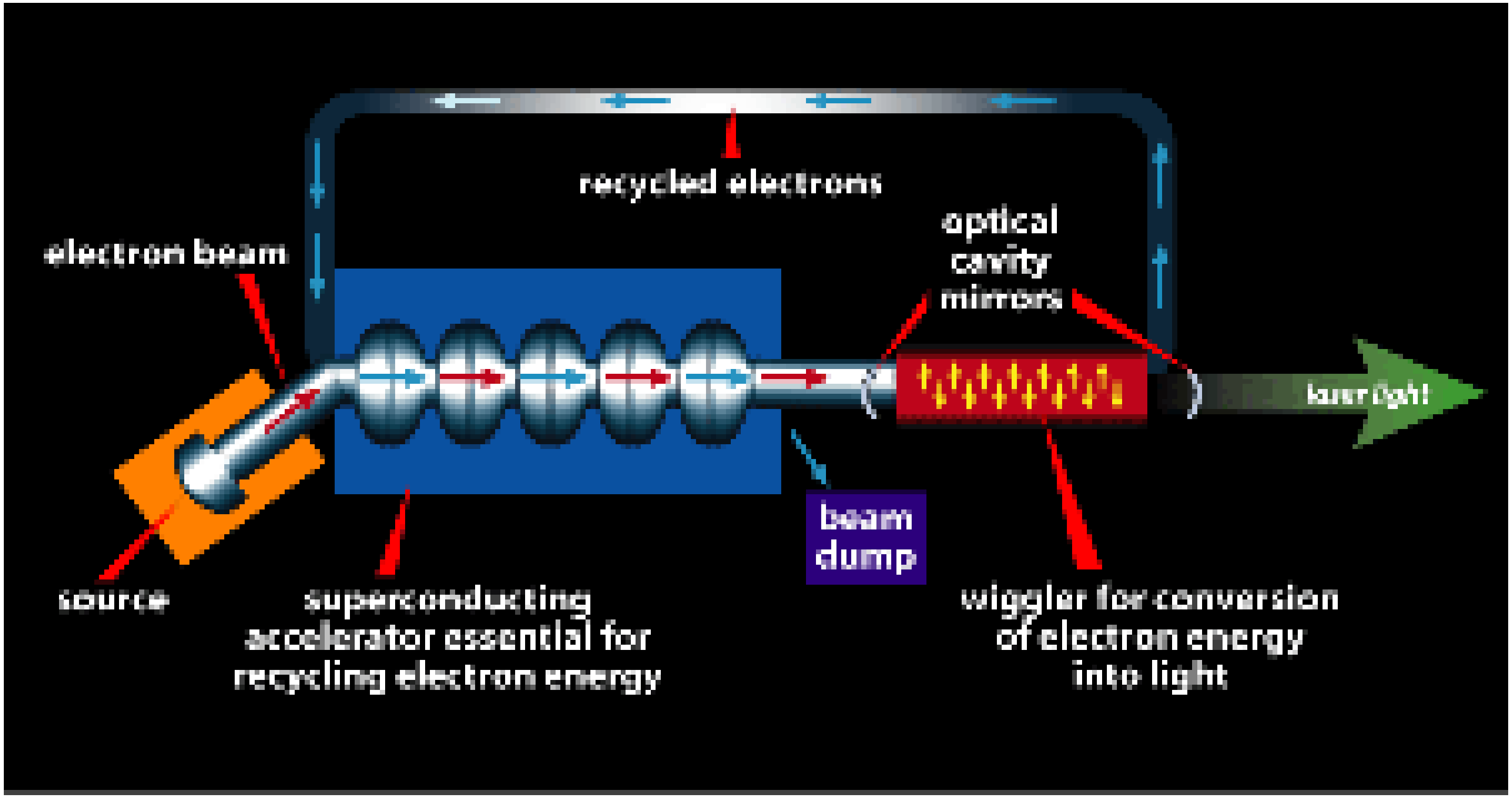
or $\delta s = \lambda_\gamma$ EM wave propagates one wavelength ahead of electron per period

Dynamics

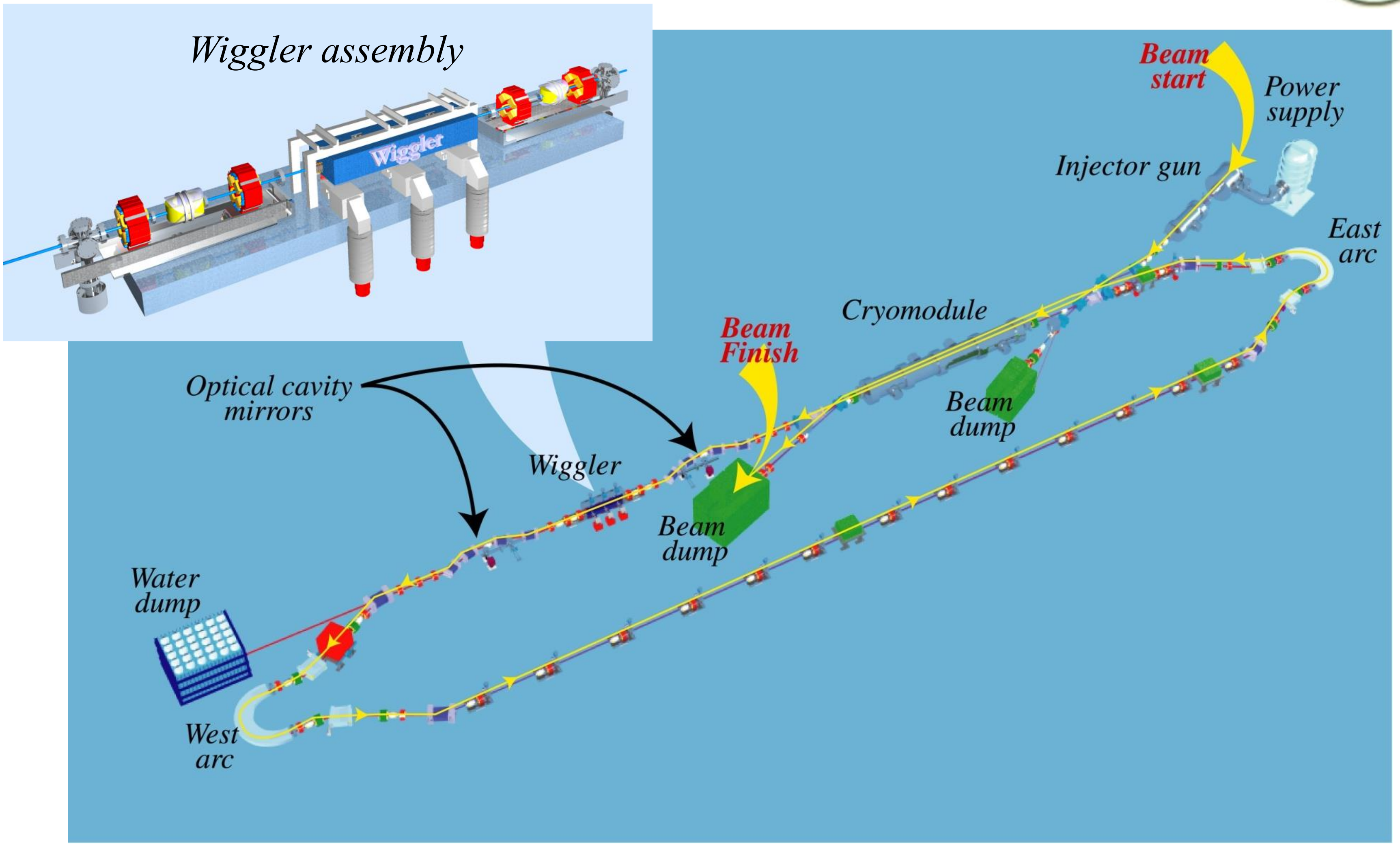


electron move constantly against external field

Energy Recovered FEL

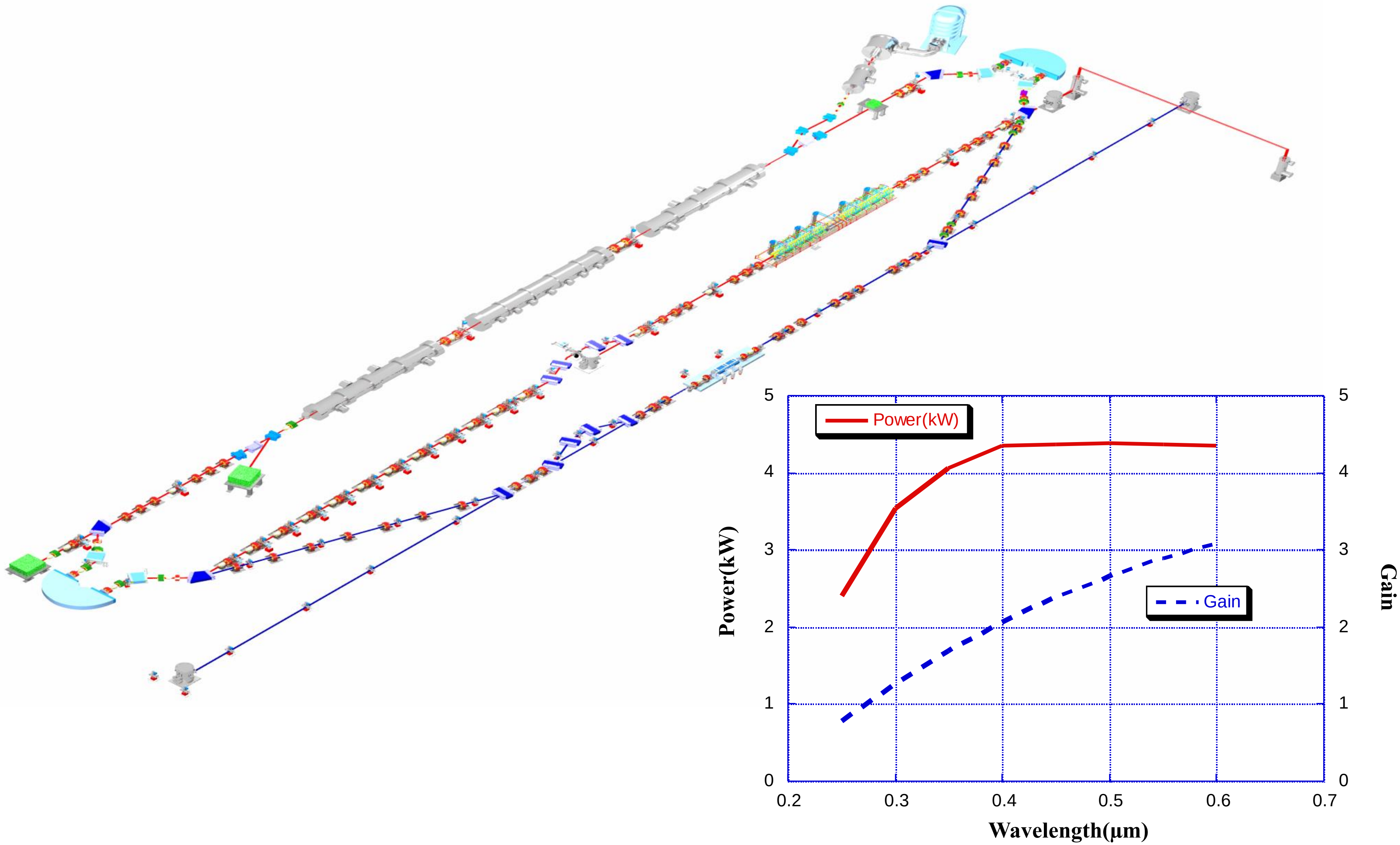


Jefferson Lab IR DEMO FEL

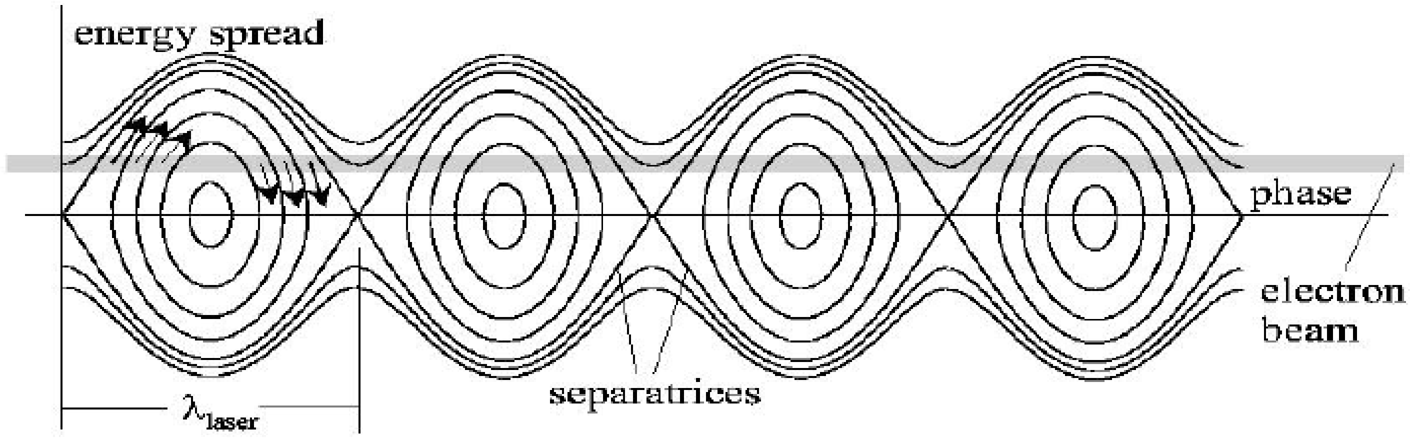


Neil, G. R., *et. al, Physical Review Letters,* 84, 622 (2000)

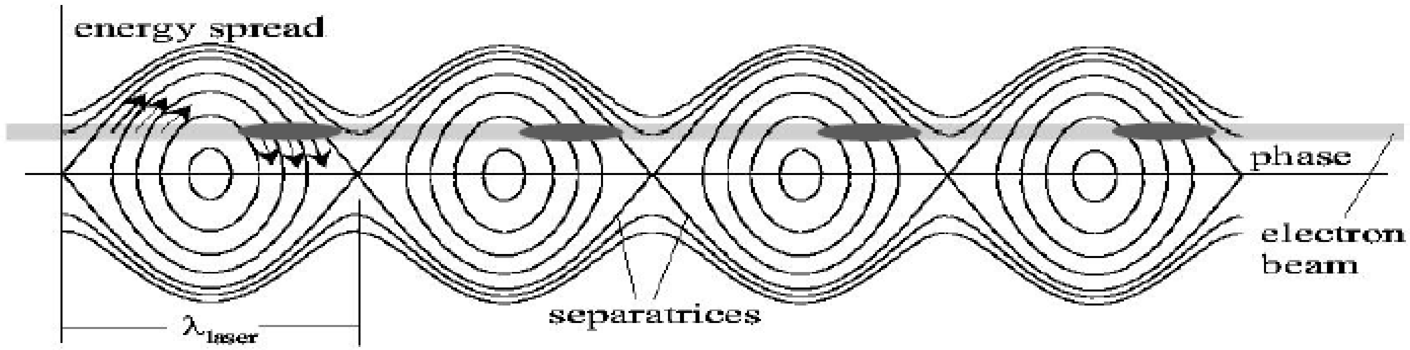
IR FEL Upgrade



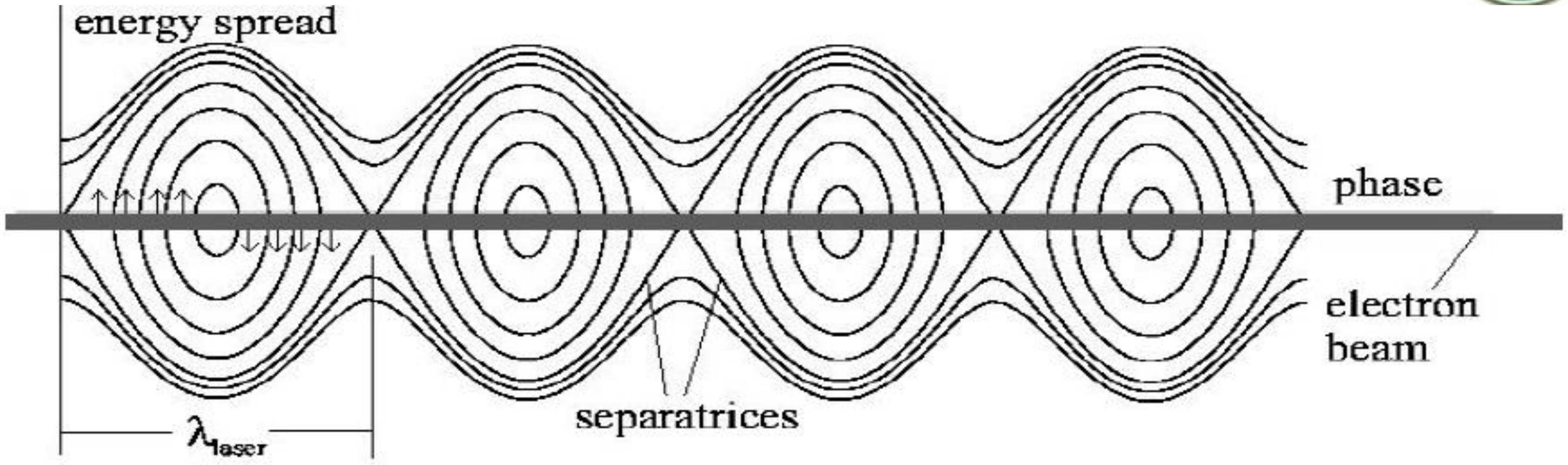
Optical Klystron



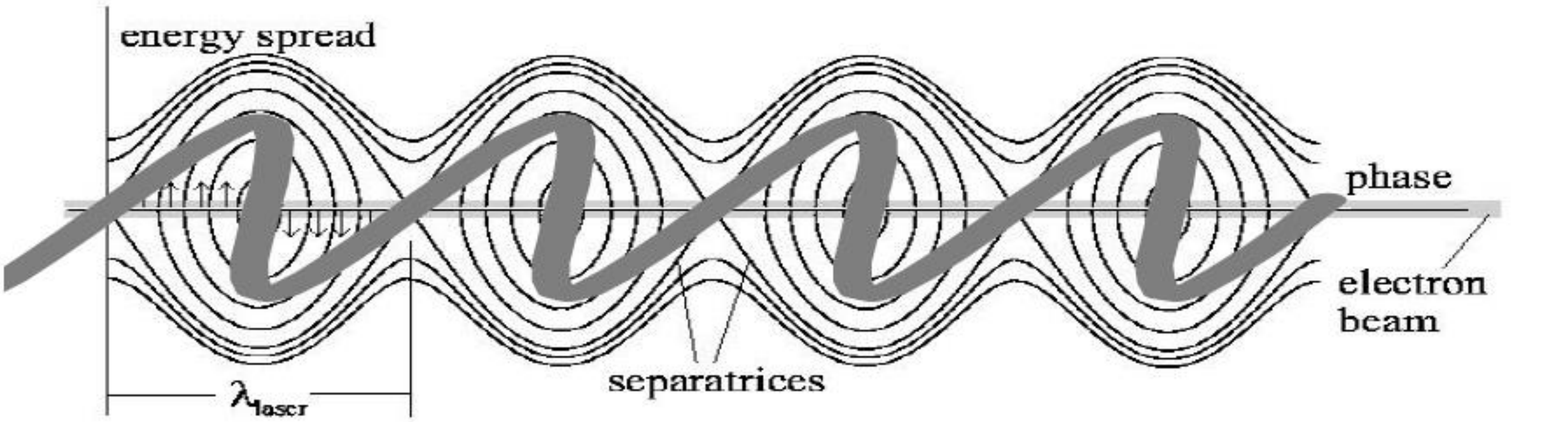
this works, but is not very efficient
bunched beam would be better



Beam Bunching

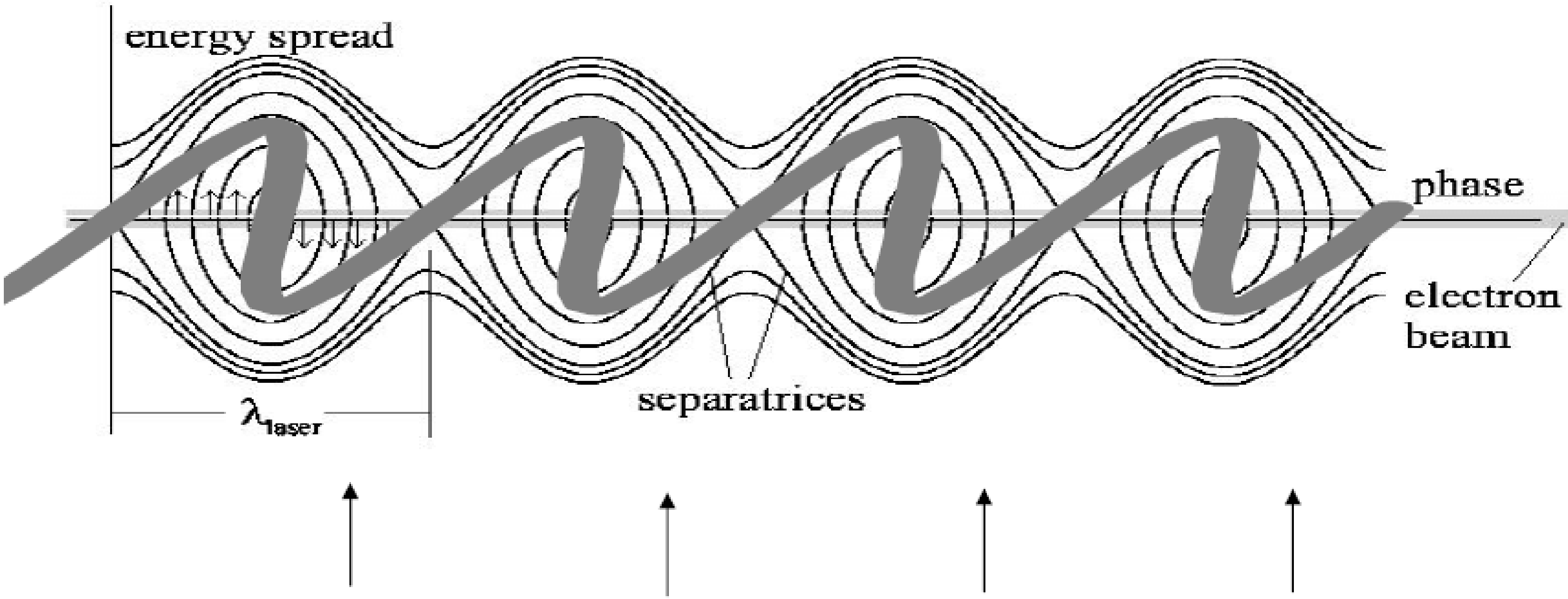


at undulator exit:



FEL action results in a bunched beam

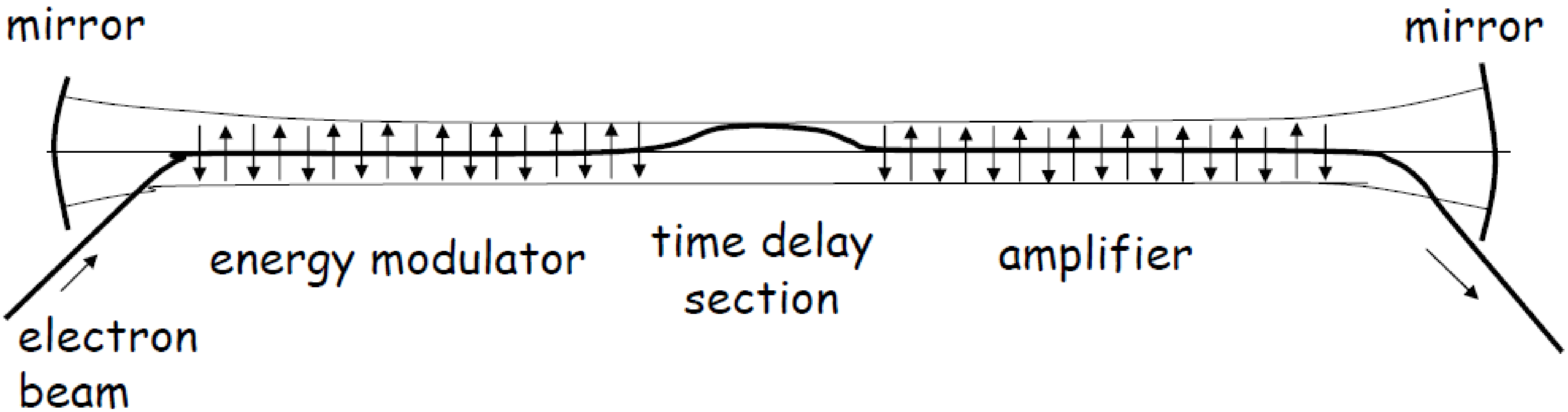
but the bunches are not at the right point



we need bunches here

need time delay section

Add Time Delay!



SASE

FEL works only for wavelength where mirrors exist

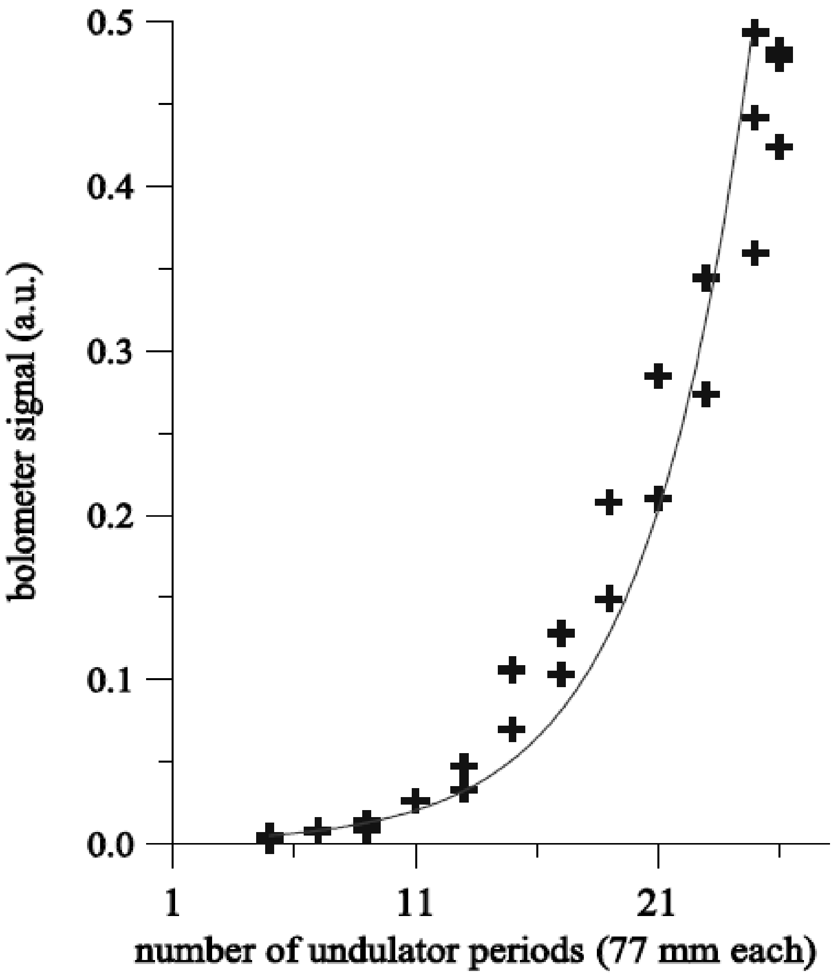
mostly visible, IR, FIR and microwaves

how about an x-ray free electron laser ?

amplification can occur only in one pass !

it can work !

53 micron SASE



How it Works

consider bunch

there is always a density fluctuation

fluctuation acts like a bunch, emitting coherent radiation

coherent radiation propagates faster than electrons

field acts back on bunch generating periodic energy variation

energy variation transforms into bunching at desired wavelength

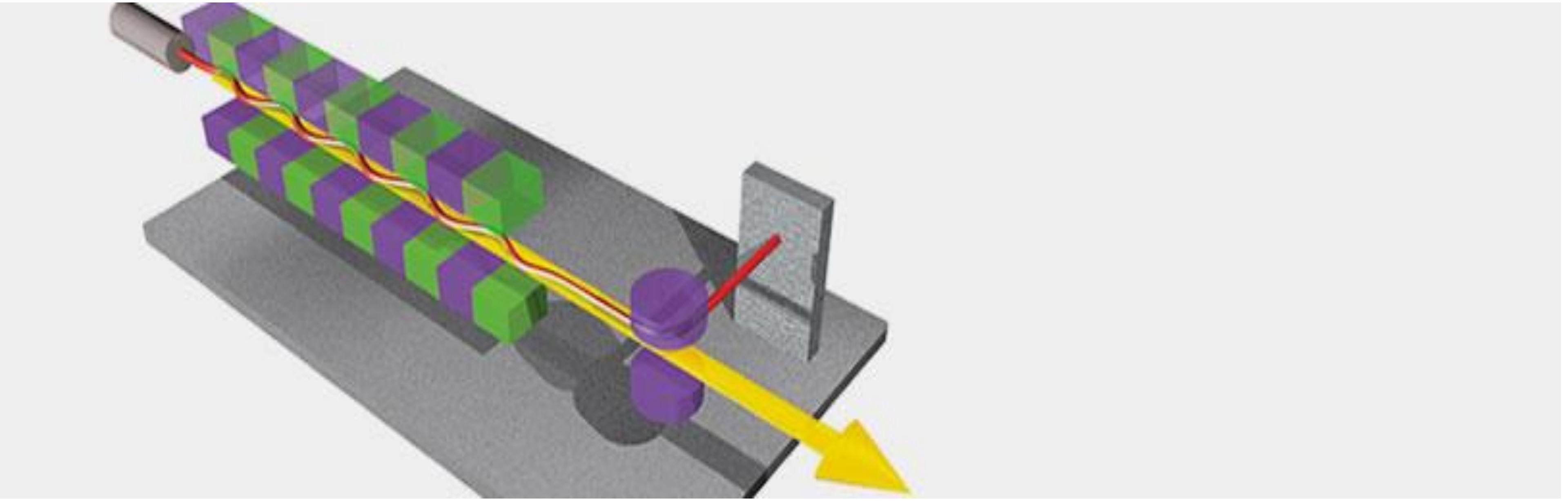
generating even more radiation growing exponentially

need long undulator: ~ 100 m (SLAC)

for 1A radiation: need electron energy about 15 GeV

need high quality, high intensity, low emittance beam

SASE FEL



XFEL 1.3 GHz Cavities

XFEL Undulator

LCLS Undulator