Fundamentals of RF Cavities

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Lecture 10
Outline

- RF Cavities
  - Cavity Basics
  - RF Properties
  - TM Type Cavities
- Types of Cavities
  - Accelerating Cavities
  - Low $\beta$ cavities
  - Deflecting/Crabbing Cavities
- Limitations in SRF Cavities
- Losses in RF Cavities
Suggested Literature


- Proceedings of the Workshops on RF Superconductivity 1981–2015 – (ww.jacow.org)

RF Cavities

RF cavities made of different materials, in different shapes and sizes

- CESR 500 MHz Cavity (Cornell)
- LEP 350 MHz 4-cell Nb on Cu
- 1300 MHz 9-cell
- Quarter Wave Cavity
- 1500 MHz 5-cell
- Triple Spoke Cavity
- Half Wave Cavity
- Single Spoke Cavity
RF Cavities

• Space enclosed by conducting walls that can sustain an infinite number of resonant electromagnetic modes

• Shape is selected so that a particular mode can efficiently transfer its energy to a charged particle

• An isolated mode can be modeled by an LRC circuit

• Lorentz force $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

• An accelerating cavity needs to provide an electric field ($E$) longitudinal with the velocity of the particle

• Magnetic fields ($H$) provide deflection but no acceleration
RF Resonator

- Simplest form of RF resonator $\rightarrow$ LC circuit

- LC circuit $\rightarrow$ Pill box cavity
  - Electric field is concentrated near axis
  - Magnetic field is concentrated at outer cylindrical wall
Cavity Basics

• Fields in an rf cavity are solution to the wave equation

\[ \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \begin{bmatrix} E \\ H \end{bmatrix} = 0 \]

• Subjected to boundary conditions:
  – No tangential electric field \( \hat{n} \times E = 0 \),
  – No normal magnetic field \( \hat{n} \cdot H = 0 \)

• Two sets of eigenmode solutions with infinite number of modes
  – TM modes \( \rightarrow \) Modes with longitudinal electric fields and no transverse magnetic fields
  – TE modes \( \rightarrow \) Modes with longitudinal magnetic fields and no transverse electric fields
TM and TE Modes in a Pill Box Cavity

**TM Modes:**

\[
E_z = E_0 \cos \left( \frac{pxz}{L} \right) J_m \left( \frac{xr_m}{R} \right) \cos(m\phi),
E_r = -E_0 \frac{pr R}{Lx_{mn}} \sin \left( \frac{pxz}{L} \right) J'_m \left( \frac{xr_m}{R} \right) \cos(m\phi),
E_\phi = E_0 \frac{mpx R^2}{rLx_{mn}^2} \sin \left( \frac{pxz}{L} \right) J_m \left( \frac{xr_m}{R} \right) \sin(m\phi),
H_z = 0,
H_r = jE_0 \frac{m\omega R^2}{r\gamma r x_{mn}^2} \cos \left( \frac{pxz}{L} \right) J_m \left( \frac{xr_m}{R} \right) \sin(m\phi),
H_\phi = jE_0 \frac{\omega R}{r\gamma r x_{mn}} \cos \left( \frac{pxz}{L} \right) J'_m \left( \frac{xr_m}{R} \right) \cos(m\phi),
\]

**TE Modes:**

\[
H_z = H_0 \sin \left( \frac{pxz}{L} \right) J_m \left( \frac{xr_m}{R} \right) \cos(m\phi),
H_r = H_0 \frac{px R}{Lx'_{mn}} \cos \left( \frac{pxz}{L} \right) J'_m \left( \frac{xr'_m}{R} \right) \cos(m\phi),
H_\phi = -H_0 \frac{mp\omega R^2}{rL(x'_{mn})^2} \cos \left( \frac{pxz}{L} \right) J_m \left( \frac{x_{mn}r}{R} \right) \sin(m\phi),
E_z = 0,
E_r = jH_0 \frac{m\omega R^2}{\gamma r(x'_{mn})^2} \sin \left( \frac{pxz}{L} \right) J_m \left( \frac{x_{mn}r}{R} \right) \sin(m\phi),
E_\phi = jH_0 \frac{\omega R}{\gamma r x'_{mn}} \sin \left( \frac{pxz}{L} \right) J'_m \left( \frac{x_{mn}r}{R} \right) \cos(m\phi),
\]

\[
\omega_{TM_{mn}} = c \sqrt{\left( \frac{x_{mn}c}{R} \right)^2 + \left( \frac{px}{L} \right)^2},
\]

\[
\omega_{TE_{mn}} = c \sqrt{\left( \frac{x'_{mn}c}{R} \right)^2 + \left( \frac{px}{L} \right)^2}.
\]

\(x_{mn}\) is the n\textsuperscript{th} root of \(J_m\)

\(x'_{mn}\) is the n\textsuperscript{th} root of \(J'_m\)
Modes in Pill Box Cavity

- **$\text{TM}_{010}$**
  - Electric field is purely longitudinal
  - Electric and magnetic fields have no angular dependence
  - Frequency depends only on radius, independent on length

- **$\text{TM}_{0\text{np}}$**
  - Monopole modes that can couple to the beam and exchange energy

- **$\text{TM}_{1\text{np}}$**
  - Dipole modes that can deflect the beam

- **TE modes**
  - No longitudinal $E$ field
  - Cannot couple to the beam
  - TE-type modes can deflect the beam
Pill Box Cavity

• Needs longitudinal electric field for acceleration
• Operated in the $\text{TM}_{010}$ mode
• Hollow right cylindrical enclosure

\[
\frac{\partial^2 E_z}{\partial^2 r} + \frac{1}{r} \frac{\partial E_z}{\partial r} = \frac{1}{c^2} \frac{\partial^2 E_z}{\partial^2 t} \quad \omega_0 = \frac{2.405c}{R}
\]

\[
E_z(r, z, t) = E_0 J_0 \left( 2.405 \frac{r}{R} \right) e^{-i\omega_0 t}
\]

\[
H_\phi(r, z, t) = -i \frac{E_0}{\mu_0 c} J_1 \left( 2.405 \frac{r}{R} \right) e^{-i\omega_0 t}
\]
**TM\(_{010}\) Mode in a Pill Box Cavity**

\[
E_r = E_\varphi = 0 \quad E_z = E_0 J_0 \left( x_{01} \frac{r}{R} \right)
\]

\[
H_r = H_z = 0 \quad H_\varphi = -i\omega \varepsilon E_0 \frac{R}{x_{01}} J_1 \left( x_{01} \frac{r}{R} \right)
\]

\[
\omega = x_{01} \frac{c}{R} \quad x_{01} = 2.405
\]

\[
R = \frac{x_{01}}{2\pi} \lambda = 0.383 \lambda
\]

- Frequency scales inversely with cavity radius
**TM\textsubscript{010} Field Profile**

- **Electric Field**
- **Magnetic Field**
- **Surface Electric Field**
- **Surface Magnetic Field**

- Peak surface electric field at end plates
- Peak surface magnetic field at outer cylindrical surface
Cavity RF Properties
Accelerating Voltage

• For efficient acceleration, choose a cavity geometry and a mode where:
  – Electric field is along the particle trajectory
  – Magnetic field is zero along the particle trajectory
  – Velocity of the electromagnetic field is matched to particle velocity

• Accelerating voltage for charged particles

\[ V_c = \left| \int_{-\infty}^{\infty} E_z(\rho = 0, z) e^{i\omega_0 z/\beta_c} dz \right| \]
Accelerating Voltage ($V_c$)

- Accelerating voltage for charged particles

$$L = \frac{\beta \lambda}{2}$$

- For the pill box cavity

$$V_c = \left| \int_{-\infty}^{\infty} E_z(\rho = 0, z) e^{i \omega_0 z / \beta c} \, dz \right|$$

$$V_c = E_0 \left| \int_{0}^{d} e^{i \omega_0 z / \beta c} \, dz \right| = E_0 d \cdot \sin \left( \frac{\omega_0 d}{2 \beta c} \right) = E_0 d \cdot T$$

$T$ is the transit time factor
Accelerating Gradient ($E_{acc}$)

- Accelerating field (gradient): Voltage gained by a particle divided by a reference length

$$E_{acc} = \frac{V_{acc}}{L}$$

- For velocity of light particles: $L = \frac{N\lambda}{2}$

  $N$ – no. of cells

- For less-than-velocity-of-light cavities ($\beta < 1$), there is no universally adopted definition of the reference length

- However multi-cell elliptical cavities with $\beta < 1$

  Length per cell $L = \frac{\beta \lambda}{2}$
Stored Energy ($U$)

- Energy density in electromagnetic field:
  \[ u = \frac{1}{2} \left( \varepsilon_0 E^2 + \mu_0 H^2 \right) \]

- Because of the sinusoidal time dependence and the 90º phase shift, the energy oscillates back and forth between the electric and magnetic field.

- Total energy content in the cavity:
  \[ U = \frac{\varepsilon_0}{2} \int_V dV \ |E|^2 = \frac{\mu_0}{2} \int_V dV \ |H|^2 \]
Power Dissipation ($P_{\text{diss}}$)

- Surface current results in power dissipation proportional to the surface resistance ($R_s$)

- Power dissipation per unit area

$$\frac{dP}{da} = \frac{\mu_0 \omega \delta}{4} |H_{||}|^2 = \frac{R_s}{2} |H_{||}|^2$$

- Total power dissipation in the cavity walls

$$P = \frac{R_s}{2} \int_A da |H_{||}|^2$$
Power Dissipation ($P_{\text{diss}}$)

\[
\frac{P}{L} \propto \frac{1}{Q} \frac{E^2 R_S}{\omega} R Q R_S
\]

- For normal conductors $\rightarrow R_S \propto \omega^{1/2}$
  - per unit length $\frac{P}{L} \propto \omega^{1/2}$
  - per unit area $\frac{P}{A} \propto \omega^{1/2}$

- For superconductors $\rightarrow R_S \propto \omega^2$
  - per unit length $\frac{P}{L} \propto \omega$
  - per unit area $\frac{P}{A} \propto \omega^2$
Quality Factor ($Q_0$)

- Measures cavity performance as to how lossy cavity material is for given stored energy

$$Q_0 \equiv \frac{\text{Energy stored in cavity}}{\text{Energy dissipated in cavity walls per radian}} = \frac{\omega_0 U}{P_{\text{diss}}}$$

$$= \omega_0 \tau_0 = \frac{\omega_0}{\Delta \omega_0}$$

$$Q_0 = \frac{\omega \mu_0}{R_s} \int_V dV |H|^2 \int_A da |H_{||}|^2$$

- For normal conducting cavities $\sim 10^4$
- For superconducting cavities $\sim 10^{10}$
Geometrical Factor ($G$)

- Geometrical factor [$\Omega$]
  - Product of the qualify factor ($Q_0$) and the surface resistance ($R_s$)
  - Independent of size and material
  - Depends only on shape of cavity and electromagnetic mode
  - For superconducting elliptical cavities $QR_s \sim 275\Omega$

\[
G = QR_s = \omega \mu_0 \frac{\int_V dV |H|^2}{\int_A da |H_\parallel|^2} = 2\pi \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{1}{\lambda} \int_V dV \frac{|H|^2}{|H_\parallel|^2} = \frac{2\pi \eta}{\lambda} \int_A dA \frac{|H|^2}{|H_\parallel|^2}
\]

$\eta \approx 377\Omega$  Impedance of vacuum
Shunt Impedance ($R_{sh}$) and $R/Q$

- Shunt impedance ($R_{sh}$) [$\Omega$]
  \[ R_{sh} \equiv \frac{V_c^2}{P_{diss}} \]

- Maximize shunt impedance to get maximum acceleration

- Note: Sometimes the shunt impedance is defined as or quoted as impedance per unit length (\(\Omega/m\))

- $R/Q$ [$\Omega$]: Measures of how much of acceleration for a given power dissipation
  \[ \frac{V_c^2}{2P_{diss}} \]
  \[ \frac{R}{Q} = \frac{V^2}{P} \frac{P}{\omega U} = \frac{E^2}{U} \frac{L^2}{\omega} \]
\( R_{sh}R_s \) and \( R/Q \)

- Optimization parameter:
  \[
  R_{sh}R_s = \frac{R_{sh}}{Q}QR_s = \frac{R}{Q}G
  \]

- \( R/Q \) and \( R_{sh}R_s \)
  - Independent of size (frequency) and material
  - Depends on mode geometry
  - Proportional to no. of cells

- In practice for elliptical cavities
  - \( R/Q \sim 100 \ \Omega \) per cell
  - \( R_{sh}R_s \sim 33,000 \ \Omega^2 \) per cell
**TM\textsubscript{010} Mode in a Pill Box Cavity**

Energy content

\[
U = \varepsilon_0 E_0^2 \frac{\pi}{2} J_1^2(x_{01})LR^2
\]

Power dissipation

\[
P = E_0^2 \frac{R_s}{\eta^2} \pi J_1^2(x_{01})(R + L)R
\]

Geometrical factor

\[
G = \eta \frac{x_{01}}{2} \frac{L}{(R + L)}
\]
TM_{010} Mode in a Pill Box Cavity

Energy Gain

\[ \Delta W = E_0 \frac{\lambda}{\pi} \sin \frac{\pi L}{\lambda} \]

Gradient

\[ E_{acc} = \frac{\Delta W}{\lambda / 2} = E_0 \frac{2}{\pi} \sin \frac{\pi L}{\lambda} \]

Shunt impedance

\[ R_{sh} = \frac{\eta^2}{R_s} \frac{1}{\pi^3 J_1^2(x_{01})} \frac{\lambda^2}{R(R + L)} \sin^2 \left( \frac{\pi L}{\lambda} \right) \]
Accelerating Cavities
Real Cavities

- Beam tubes reduce the electric field on axis
  - Gradient decreases
  - Peak fields increase
  - $R/Q$ decreases
Pill Box to Elliptical Cavities

TM010

$\ f = 1323 \text{ MHz}$
Single Cell Cavities

Electric field high at iris

Magnetic field high at equator

- Important parameters: $E_p/E_{acc}$ and $B_p/E_{acc}$
- Must minimize the ratios as smaller as possible
## Single Cell Cavities

![Superconducting B-Factory Cell Shape](image)

### Table

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Cornell SC 500 MHz</th>
<th>Pillbox</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$</td>
<td>270 Ω</td>
<td>257 Ω</td>
</tr>
<tr>
<td>$R_a/Q_0$</td>
<td>88 Ω/cell</td>
<td>196 Ω/cell</td>
</tr>
<tr>
<td>$E_{pk}/E_{acc}$</td>
<td>2.5</td>
<td>1.6</td>
</tr>
<tr>
<td>$H_{pk}/E_{acc}$</td>
<td>52 Oe/MV/m</td>
<td>30.5 Oe/(MV/m)</td>
</tr>
</tbody>
</table>
Cell Shape Design

- What is the purpose of the cavity?
- What EM parameters should be optimized to meet the design specs?

The “perfect” shape does not exist, it all depends on your application

- Beam aperture
- Peak surface field ratios – $E_p/E_{acc}$, $B_p/E_{acc}$
- Shunt impedance – $R_{sh}R_s$
- Higher Order Mode (HOM) extraction
Example: CEBAF Upgrade

- “High Gradient” shape: lowest $E_p/E_{acc}$
- “Low Loss” shape: lowest cryogenic losses $R_{sh}R_s = G(R/Q)$
The field emission is not a hard limit in the performance of sc cavities if the surface preparation is done in the right way.

Unlikely this, magnetic flux on the wall limits performance of a sc cavity ($Q_0$ decreases or/and quench). Hard limit $\sim 180$ mT for Nb.

$B_{\text{peak}} / E_{\text{acc}}$ should be low.

1. Cavities may operate at higher gradients.
2. Cavities may operate at lower cryogenic load.

$$\frac{P_{\text{diss}}}{V_{\text{acc}}^2} \equiv \frac{R_s}{G \cdot (R / Q)}$$
New Shapes for ILC

\[ f = 1300 \text{ MHz} \]

<table>
<thead>
<tr>
<th></th>
<th>( r_{\text{iris}} ) [mm]</th>
<th>35</th>
<th>30</th>
<th>33</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_{cc} ) [%]</td>
<td>1.9</td>
<td>1.52</td>
<td>1.8</td>
<td></td>
</tr>
<tr>
<td>( E_{\text{peak}}/E_{\text{acc}} )</td>
<td>1.98</td>
<td>2.36</td>
<td>2.21</td>
<td></td>
</tr>
<tr>
<td>( B_{\text{peak}}/E_{\text{acc}} ) [mT/(MV/m)]</td>
<td>4.15</td>
<td>3.61</td>
<td>3.76</td>
<td></td>
</tr>
<tr>
<td>( R/Q ) [Ω]</td>
<td>113.8</td>
<td>133.7</td>
<td>126.8</td>
<td></td>
</tr>
<tr>
<td>( G ) [Ω]</td>
<td>271</td>
<td>284</td>
<td>277</td>
<td></td>
</tr>
<tr>
<td>( R/Q \times G ) [Ω*Ω]</td>
<td>30840</td>
<td>37970</td>
<td>35123</td>
<td></td>
</tr>
</tbody>
</table>
RF Simulation Codes for Cavity Design

The solution to 2D (or 3D) Helmholtz equation can be analytically found only for very few geometries (pillbox, spherical resonators or rectangular resonator).

We need numerical methods:

\[ (\nabla^2 + \omega^2 \varepsilon \mu) A = 0 \]

- **Approximating operator** (Finite Difference Methods)
- **Approximating function** (Finite Element Methods)

- 2D is fast and allows to define geometry of a cylindrical symmetric body (inner and end-cells) of the cavity.

- 3D is much more time consuming but necessary for modeling of full equipped cavity with FPC and HOM couplers and if needed to model fabrication errors. Also coupling strength for FPC and damping of HOMs can be modeled only 3D.
SUPERFISH

- Free, 2D finite-difference code to design cylindrically symmetric structures (monopole modes only)
- Use symmetry planes to reduce number of mesh points

http://laacg1.lanl.gov/laacg/services/download_sf.phtml
CST Microwave Studio

- Expensive, 3D finite-element code, used to design complex RF structure
- Runs on PC
- Perfect boundary approximation

http://www.cst.com/Content/Products/MWS/Overview.aspx
Omega3P – ACE3P (SLAC Code Suite)

- SLAC, 3D code, high-order Parallel Finite Element (PFE) method
- Runs on Linux
- Tetrahedral conformal mesh
- High order finite elements (basis order $p = 1 – 6$)
- Separate software for user interface (Cubit), and visualization and post processing (ParaView)
Cell Shape Parametrization

- Full parametric model of the cavity in terms of 7 meaningful geometrical parameters:
  - Ellipse ratio at the equator \((R=B/A)\) ruled by mechanics
  - Ellipse ratio at the iris \((r=b/a)\)
  - \(E_{peak}\)
  - Side wall inclination \((\alpha)\) and position \((d)\)
  - \(E_{peak}\) vs. \(B_{peak}\) tradeoff and coupling \(k_{cc}\)
  - Cavity iris radius \(R_{iris}\)
  - Coupling \(k_{cc}\)
  - Half-cell Length \(L/2=\lambda\beta/4\)
  - \(\beta\)
  - Cavity radius \(D\) used for frequency tuning

- Behavior of all EM and mechanical properties has been found as a function of the above parameters
Reducing Peak Surface Fields

• “Rule of thumb” for Optimizing Peak Surface Fields

Add “magnetic volume” at the equator to reduce $B_{\text{peak}}$

Add “electric volume” at the iris to reduce $E_{\text{peak}}$
Multi-Cell Cavities

Single-cell is attractive from the RF-point of view:
- Easier to manage HOM damping
- No field flatness problem
- Input coupler transfers less power
- Easy for cleaning and preparation

- But it is expensive to base even a small linear accelerator on the single cell. We do it only for very high beam current machines.

A multi-cell structure is less expensive and offers higher real-estate gradient but:
- Field flatness (stored energy) in cells becomes sensitive to frequency errors of individual cells
- Other problems arise: HOM trapping…
Cell to Cell Coupling

The normalized difference between these frequencies is a measure of the energy flow via the coupling region.

\[ k_{cc} = \frac{\omega_\pi - \omega_0}{\omega_\pi + \omega_0} \times \frac{1}{2} \]
Pros and Cons of Multi-Cell Cavities

- Cost of accelerators are lower (less auxiliaries: He vessels, tuners, fundamental power couplers, control electronics)
- Higher real-estate gradient (better fill factor)

- Field flatness vs. N (N – no. of cells)
- HOM trapping vs. N
- Power capability of fundamental power couplers vs. N
- Chemical treatment and final preparation become more complicated
- The worst performing cell limits whole multi-cell structure
Multi-Cell Cavities

\[ k = \frac{C}{C_k} \quad C_b = C_k / 2 \]

Mode frequencies:

\[
\frac{\omega_m^2}{\omega_0^2} = 1 + 2k \left(1 - \cos \frac{\pi m}{n}\right)
\]

\[
\frac{\omega_n - \omega_{n-1}}{\omega_0} = k \left(1 - \cos \frac{\pi}{n}\right) = \frac{k}{2} \left(\frac{\pi}{n}\right)^2
\]

\( \omega_0 \) – accelerating mode frequency

Voltages in cells:

\[
V_j^m = \sin \left(\pi m \frac{2j-1}{2n}\right)
\]
Coupling of TM$_{010}$ modes of the individual cells via the iris (primarily electric field) causes them to split into a passband of closely spaced modes equal in number to the number of cells.

$$\omega_0 (1 + 4k)^{1/2}$$

9-cell cavity
Cell Excitations in Pass-Band Modes
Field Flatness in Multi-Cell Cavities

- Geometrical differences between cells causes a mixing of the eigenmodes

- Sensitivity to mechanical deformation depends on mode spacing

\[
\frac{\omega_n - \omega_{n-1}}{\omega_0} = k \left( 1 - \cos \frac{\pi}{n} \right) = \frac{k}{2} \left( \frac{\pi}{n} \right)^2
\]
Low Beta Accelerating Structures – TEM Class Cavities
Classification of Structures

- Two main types of structure geometries
  - TEM class (QW, HW, Spoke)
  - TM class (Elliptical)
Low $\beta$ Cavities

- Low $\beta$ cavities: Cavities that accelerate particles with $\beta < 1$ efficiently

- Increased needs for reduced-beta ($\beta < 1$) SRF cavity especially in CW machines or high duty pulsed machine (duty > 10%)

- Reduced beta Elliptical multi-cell SRF cavity
  - For CW, prototyping by several R&D groups have demonstrated as low as $\beta=0.47$
  - For pulsed, SNS $\beta=0.61, 0.81$ cavities & ESS

- Elliptical cavity has intrinsic problem as $\beta$ goes down
  - Mechanical problem, multipacting, low rf efficiency
# Applications of Low $\beta$ Cavities

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<th>Medium/Low Current</th>
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<tr>
<td><strong>CW</strong></td>
<td></td>
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<tr>
<td>Accelerator driven systems</td>
<td>Production of radioactive ions</td>
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<tr>
<td>- Waste transmutation</td>
<td>Nuclear Structure</td>
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<tr>
<td>- Energy production</td>
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<tr>
<td>Beam: p, H, d</td>
<td>Beam: p to U</td>
</tr>
<tr>
<td><strong>Pulsed</strong></td>
<td></td>
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<tr>
<td>Pulsed spallation sources</td>
<td></td>
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<tr>
<td>Beam: p, H-</td>
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## Technical Issues and Challenges

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<th>Medium/Low Current</th>
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<tr>
<td><strong>CW</strong></td>
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<tr>
<td>- Beam losses (~ 1 W/m)</td>
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<tr>
<td>- Activation</td>
<td>- Activation</td>
</tr>
<tr>
<td>- High cw rf power</td>
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<tr>
<td>- Higher order modes</td>
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<tr>
<td>- Cryogenics losses</td>
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<tr>
<td><strong>Pulsed</strong></td>
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<td>- Higher order modes</td>
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<tr>
<td>- High peak rf power</td>
<td>- High peak rf power</td>
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<tr>
<td>- Dynamic Lorentz detuning</td>
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### Design Considerations

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<td>CW</td>
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<tr>
<td>- Cavities with high acceptance</td>
<td>- Cavities with low sensitivity to vibration</td>
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<tr>
<td>- Development of high cw power couplers</td>
<td>- Development of microphonics compensation</td>
</tr>
<tr>
<td>- Extraction of HOM power</td>
<td>- Cavities with high shunt impedance</td>
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<tr>
<td>- Cavities with high shunt impedance</td>
<td>- Cavities with large velocity acceptance (few cells)</td>
</tr>
<tr>
<td>Pulsed</td>
<td>- Cavities with large beam acceptance (low frequency, small frequency transitions)</td>
</tr>
<tr>
<td>- Cavities with high acceptance</td>
<td>Note:</td>
</tr>
<tr>
<td>- Development of high peak power couplers</td>
<td>- Large beam acceptance</td>
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<tr>
<td>- Extraction of HOM power</td>
<td>- Large aperture (transverse acceptance)</td>
</tr>
<tr>
<td>- Development of active compensation of dynamic Lorentz detuning</td>
<td>- Low frequency (longitudinal acceptance)</td>
</tr>
</tbody>
</table>

**Note:**

- Large beam acceptance
- Large aperture (transverse acceptance)
- Low frequency (longitudinal acceptance)
Basic Geometries

- Resonant transmission lines
  - $\lambda/4$
    - Quarter wave
    - Split ring
    - Twin quarter wave
    - Lollipop
  - $\lambda/2$
    - Coaxial half wave
    - Spoke
    - H-type

- TM type
  - Elliptical
  - Reentrant

- Other
  - Alvarez
  - Slotted iris
Low $\beta$ Cavities

- Many different shapes and sizes

ANL cavities for RIA

QW
HW
Spoke

Half-wave cavity

115 MHz split-ring cavity,
172.5 MHz $\beta = 0.19$ “lollipop” cavity

RFQ

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Transit Time Factor

- The transit time factor is the ratio of the acceleration voltage to the (non-physical) voltage a particle with infinite velocity would see.

- Energy gain ($\Delta W$):
  \[ \Delta W = q \int_{-\infty}^{+\infty} E(z) \cos(\omega t + \phi) \, dz \]

- Assuming constant velocity
  \[ \Delta W = q \cos \phi \ \Delta W_0 \ \ T(\beta) \quad \Delta W_0 = \Theta \int_{-\infty}^{+\infty} |E(z)| \, dz \]

- Transit time factor:
  \[ \Theta = \frac{\max \int_{-\infty}^{+\infty} E(z) \cos \left( \frac{\omega z}{\beta c} \right) \, dz}{\int_{-\infty}^{+\infty} |E(z)| \, dz} \]
Velocity Acceptance

- Velocity acceptance

\[ T(\beta) = \frac{\int_{-\infty}^{+\infty} E(z) \cos\left(\frac{\omega z}{\beta c}\right) \, dz}{\text{Max} \int_{-\infty}^{+\infty} E(z) \cos\left(\frac{\omega z}{\beta c}\right) \, dz} \]

- Lower the velocity of the particle or cavity \( \beta \)
  - Faster the velocity of the particle will change
  - Narrower the velocity range of a particular cavity
  - Smaller the number of cavities of that \( \beta \)
  - More important: Particle achieve design velocity

Velocity acceptance for sinusoidal field profile
Quarter–Wave Resonator

- **$b$**: radius of outer conductor
- **$r(z)$**: radius of center conductor
- **$r_0$**: radius of center conductor at shorting plate
- **$\rho = r/b$**: normalized radius of center conductor
- **$z$**: distance from shorting plate
- **$\zeta = 4z/\lambda$**: normalized distance from shorting plate
- **$\eta = \sqrt{\mu_0/\varepsilon_0} \approx 377\Omega$**: impedance of vacuum
Quarter–Wave Resonator

Capacitively loaded $\lambda/4$ transmission line

- Capacitance per unit length:
  \[ C = \frac{2\pi\varepsilon_0}{\ln(b/r_0)} = \frac{2\pi\varepsilon_0}{\ln(1/\rho_0)}. \]

- Inductance per unit length:
  \[ L = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{r_0}\right) = \frac{\mu_0}{2\pi} \ln\left(\frac{1}{\rho_0}\right). \]

- Current along the center conductor:
  \[ I = I_0 \cos\left(\frac{2\pi}{\lambda} z\right) = I_0 \cos\left(\frac{\pi}{2} \zeta\right). \]

- Voltage along the center conductor:
  \[ V = V_0 \sin\left(\frac{2\pi}{\lambda} z\right) = V_0 \sin\left(\frac{\pi}{2} \zeta\right). \]

- Transmission line impedance:
  \[ Z_0 = \frac{V_0}{I_0} = \frac{\eta}{2\pi} \ln\left(\frac{1}{\rho_0}\right). \]

- Loading capacitance:
  \[ \Gamma = \lambda\varepsilon_0 \frac{\cot\left(\frac{2\pi}{\lambda} z\right)}{\ln(b/r_0)} = \lambda\varepsilon_0 \frac{\cot\left(\frac{\pi}{2} \zeta\right)}{\ln(1/\rho_0)}. \]

The transmission line can be shorter than $\lambda/4$ and still resonate at the right frequency if it is terminated by the appropriate loading capacitance $\Gamma$.
Quarter–Wave Resonator

Cavity parameters:
– Peak magnetic field
\[ \frac{V_p}{b} = \left\{ \frac{\eta H}{c B} \right\} \rho_0 \ln \left( \frac{1}{\rho_0} \right) \sin \left( \frac{\pi}{2} \zeta \right) \left\{ \text{m, A/m} \right\} \left\{ \text{cm, G} \right\} \]

\( V_p \): Voltage across loading capacitance
– Power dissipation
\[ P = V_p^2 \frac{\pi R_s}{8 \eta^2 b} \frac{\lambda}{\ln^2 \rho_0} \frac{1+1/\rho_0}{\zeta + \frac{1}{\pi} \sin \pi \zeta} \propto \frac{R_s}{\eta^2} E^2 \beta \lambda^2 \]

Ignore losses in the end plate
– Stored energy
\[ U = V_p^2 \frac{\pi \varepsilon_0}{8} \frac{\lambda}{\ln(1/\rho_0)} \frac{1}{\sin^2 \frac{\pi}{2} \zeta} \zeta + \frac{1}{\pi} \sin \pi \zeta \propto \varepsilon_0 E^2 \beta^2 \lambda^3 \]

– Geometrical factor
\[ G = Q R_s = 2 \pi \eta \frac{b \ln(1/\rho_0)}{\lambda} \frac{1+1/\rho_0}{\zeta + \frac{1}{\pi} \sin \pi \zeta} \propto \eta \beta \]

– Shunt impedance
\[ R_{sh} = \frac{\eta^2}{\lambda} = \frac{32 b}{\pi} \frac{\ln^2 \rho_0}{\lambda^{1+1/\rho_0}} \frac{\sin^2 \frac{\pi}{2} \zeta}{\zeta + \frac{1}{\pi} \sin \pi \zeta} \]

\[ R_{sh} \propto \eta^2 \beta \]

– \( R/Q \)
\[ \frac{R_{sh}}{Q} = \frac{16}{\pi^2} \eta \ln(1/\rho_0) \frac{\sin^2 \frac{\pi}{2} \zeta}{\zeta + \frac{1}{\pi} \sin \pi \zeta} \propto \eta \]

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Half–Wave Resonator
Half–Wave Resonator

- Capacitance per unit length
  \[ C = \frac{2\pi \varepsilon_0}{\ln\left(\frac{b}{a}\right)} - \frac{2\pi \varepsilon_0}{\ln\left(\frac{1}{\rho_0}\right)} \]

- Inductance per unit length
  \[ L = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{r_0}\right) = \frac{\mu_0}{2\pi} \ln\left(\frac{1}{\rho_0}\right) \]

- Center conductor voltage
  \[ V(z) = V_0 \cos\left(\frac{2\pi}{\lambda} z\right) \]

- Center conductor current
  \[ I(z) = I_0 \sin\left(\frac{2\pi}{\lambda} z\right) \]

- Line impedance
  \[ Z_0 = \frac{V_0}{I_0} = \frac{\eta}{2\pi} \ln\left(\frac{1}{\rho_0}\right) \]
Half–Wave Resonator

Cavity parameters:

- Peak magnetic field

\[
\frac{V_p}{b} = \begin{cases} \eta & H \\ c & B \\ \frac{300}{B} \end{cases} \rho_0 \ln\left(\frac{1}{\rho_0}\right) \begin{cases} m, \text{ Av/m} \\ m, \text{ T} \\ \text{cm, G} \end{cases}
\]

\(V_p\): Voltage across loading capacitance

- Power dissipation

\[
P = V_p^2 \frac{\pi}{4} \frac{R_s}{\eta^2} \frac{\lambda}{b} \frac{1+1/\rho_0}{\ln^2 \rho_0} \propto \frac{R_s}{\eta^2} E^2 \beta \lambda^2
\]

Ignore losses in the end plates

- Stored energy

\[
U = V_p^2 \frac{\pi \varepsilon_0}{4} \frac{\lambda}{\ln(1/\rho_0)} \propto \varepsilon_0 E^2 \beta^2 \lambda^3
\]

- Geometrical factor

\[
G = QR_s = 2\pi \eta \frac{b}{\lambda} \ln(1/\rho_0) \propto \eta \beta
\]

- Shunt impedance

\[
R_{sh} = \frac{\eta^2}{R_s} \frac{16}{\pi} \frac{b}{\lambda} \frac{1}{1+1/\rho_0}
\]

\[
R_{sh} R_s \propto \eta^2 \beta
\]

- \(R/Q\)

\[
\frac{R_{sh}}{Q} = \frac{8}{\pi^2} \eta \ln(1/\rho_0) \propto \eta
\]
There have been extensive efforts for design optimization especially to reduce the ratios of

\[ \frac{E_p}{E_{\text{acc}}} \text{ and } \frac{B_p}{E_{\text{acc}}} \]

- Controlling A/B (\( \frac{E_p}{E_{\text{acc}}} \)) and C/D (\( \frac{B_p}{E_{\text{acc}}} \)) \( \rightarrow \) Shape optimization
- Flat contacting surface at spoke base will also help in minimization of \( \frac{B_p}{E_{\text{acc}}} \)
- For these cavities:
  - Calculations agree well \( \rightarrow \) \( E_p/E_{\text{acc}} \approx 3, B_p/E_{\text{acc}} \approx (7\sim8) \text{ mT/(MV/m)} \)
  - Though it is tricky to obtain precise surface field information from the 3D simulation

325 MHz, \( \beta = 0.17 \) (FNAL)
• \( E_p/E_a \sim 3.3 \), independent of \( \beta \)
• \( B/E_a \sim 8 \text{ mT}/(\text{MV}/\text{m}) \), independent of \( \beta \)
Spoke Resonators

- $QR_s \sim 200 \beta \ \Omega$
- $R_{sh/Q} \sim 205 \ \Omega$, independent of $\beta$

Geometrical factor

$QR_s (\Omega)$

$R_{sh/Q}$ per cell (\Omega)

Lines: Elliptical
Squares: Spoke
Spoke Resonators

- $R_{sh} R_s \sim 40000 \beta \ [\Omega^2]$
- $U/E^2 \sim 200 \beta^2 \ [\text{mJ}]$

![Graph of $R_{sh} R_s$ vs Beta](image1)

![Graph of $U/E^2$ vs Beta](image2)

Lines: Elliptical

Squares: Spoke
Features of Spoke Cavities

• **Small Size**
  – About half of TM cavity of same frequency

• **Allows low frequency at reasonable size**
  – Possibility of 4.2 K operation
  – High longitudinal acceptance

• **Fewer number of cells**
  – Wider velocity acceptance

350 MHz, $\beta = 0.45$
Deflecting / Crabbing Cavities
Deflecting Cavities

- Complete bunch is deflected with the transverse kick applied at the center of the bunch.

- RF frequency \( f_d \) = 499 MHz
- Beam energy \( E \) = 11.023 GeV
- Deflecting angle \( \theta \)
- Transverse voltage \( V_t \)

\[
\begin{align*}
\theta &= \arctan \left[ \frac{eV_t}{E_0} \right] \approx \frac{eV_t}{E_0} \\
V_t &= E_0 [eV] \theta [\text{rad}]
\end{align*}
\]
Crabbing Cavities

- Luminosity with no crabbing system:
  \[ \mathcal{L} = \frac{N_1 N_2 f_c N_b}{4 \pi \sigma_x \sigma_y} F_c = \frac{N_1 N_2 f_c N_b}{4 \pi \sigma_x \sigma_y} \frac{1}{\sqrt{1 + \left( \frac{\sigma_z \theta_c}{2 \sigma_x} \right)^2}} \]

- Luminosity with crabbing system:
  - For head-on collision
    \[ \mathcal{L} = \frac{N_1 N_2 f_c N_b}{4 \pi \sigma_x \sigma_y} \]

- Transverse voltage
  \[ V_t = \frac{c E_0 \tan(\theta_c/2)}{\omega \sqrt{\beta_{crab} \beta^*} \sin(\psi_{cc-\text{ip}}^x)} \]
Both deflecting and crabbing resonant cavities are required to generate a transverse momentum.

Can be produced by either or by both transverse electric ($E_t$) and magnetic ($B_t$) fields.

Lorentz force:

$$\vec{p}_t = \int_{-\infty}^{\infty} \vec{F}_t \, dt = \frac{q}{v} \int_{-\infty}^{+\infty} \left[ \vec{E}_t + j(\vec{v} \times \vec{B}_t) \right] \, dz$$

Types of designs:
- TM-type designs $\rightarrow$ Main contribution from $B_t$
- TE-type designs $\rightarrow$ Main contribution from $E_t$
- TEM-type designs $\rightarrow$ Contribution from both $E_t$ and $B_t$
Panofsky–Wenzel Theorem

- For particles moving virtually at $v=c$, the integrated transverse force (kick) can be determined from the transverse variation of the integrated longitudinal force
  \[ j \frac{\omega}{c} \vec{F}_t = \nabla_t \vec{F}_z \]

- Transverse momentum is related to the gradient of the longitudinal electric field along the beam axis
  \[ \vec{p}_t = -i \frac{q}{\omega} \int_{-\infty}^{+\infty} \nabla_t E_z \, dz \]
  \[ p_t = -i \frac{q}{\omega} \lim_{r_0 \to 0} \frac{1}{r_0} \int_{-\infty}^{+\infty} [E_z(r_0, z) - E_z(0, z)] \, dz \]

- According to the theorem:
  - In a pure TE mode the contribution to the deflection from the magnetic field is completely cancelled by the contribution from the electric field

Deflecting/Crabbing Cavities

- KEK TM_{110} crabbing cavity
- Superconducting 4-rod cavity
- BNL double quarter-wave cavity
Deflecting/Crabbing Cavities
LHC High Luminosity Upgrade

SPS Test Prototype Module before LS2

SM18 Area
Cavity cold testing, clean room facilities & cryomodule qualification

CMS LHC Point 5 (horizontal)
4 cavities / IP side/ beam to be installed during LS3

ATLAS LHC Point 1 (vertical)
4 cavities / IP side/ beam to be installed during LS3
Cavity Limitations
Cavity Performance

![Graph showing cavity performance with different stages such as Residual losses, Ideal, Quench, Multipacting, Field emission, Thermal breakdown, and RF Processing.](image)
Multipacting

- Multipacting condition – A large amount of secondary electrons are emitted from the cavity surface by the incident primary electrons
Multipacting

- Resonant condition
  - The secondary electrons have localized and sustainable resonant trajectories with the cavity rf fields
  - Impact energies corresponds to a secondary emission yield (SEY) greater than one

For Nb: \( E_I > 150 \text{ eV} \quad E_{II} < 2000 \text{ eV} \)
Field Emission

- Characterized by exponential decay in $Q_0$

- Exponential increase of losses due to acceleration of electron field emission

- With the increasing rf field the field emitters generate an electron current leading to excessive heating and x-rays produced by bremsstrahlung

- It is a general difficulty in accelerating structures, but does not present an ultimate fundamental limit to the maximum surface electric field

- Main cause of FE is particulate contamination

  Temperature map shows line heating along the longitude at the location of the emitter
Field Emission

- Fowler and Nordheim (FN) showed that when the work function barrier at the metal surface is lowered by an applied surface electric field, electrons can tunnel through

- Electrostatic potential of the metal-vacuum interface

- Tunneling current density

\[ J = \frac{1.54 \times 10^{-6} E^2}{\Phi} \exp \left( -\frac{6.83 \times 10^9 \Phi^{3/2}}{E} \right) \]

- $J$: Current density (A/m$^2$)
- $E$: Electric field (MV/m)
- $\Phi$: Work function (eV)
Field Emission

- Modified Fowler Nordheim equation for rf fields

\[ J = k \frac{1.54 \times 10^{-6} (\beta E)^{5/2}}{\Phi} \exp \left( - \frac{6.83 \times 10^{9} \Phi^{3/2}}{\beta E} \right) \]

- Enhancement factor (10s to 100s)
- Effective emitting surface

- Field emitters

- FE can be prevented by proper surface preparation and contamination control

- Possible to reduce if not completely eliminate FE using CW RF processing, High-power Pulsed Processing (HPP) and/or Helium processing
Ponderomotive Effects

- Ponderomotive effects: changes in frequency caused by the electromagnetic field (radiation pressure)
  - Static Lorentz detuning (CW operation)
  - Dynamic Lorentz detuning (pulsed operation)
- Microphonics: changes in frequency caused by connections to the external world
  - Vibrations
  - Pressure fluctuations
- Note: The two are not completely independent. When phase and amplitude feedbacks are active, the ponderomotive effects can change the response to external disturbances
Lorentz Force Detuning

- Electromagnetic fields in a cavity exert Lorentz forces on the cavity wall. The force per unit area (radiation pressure) is given by

\[ P = \frac{\mu_0 H_s^2 - \varepsilon_0 E_s^2}{4} \]

- Residual deformation of the cavity shape shifts the resonant frequency

\[ \frac{\Delta f_L}{f} \approx \frac{1}{4U} \int_{\Delta V} (\mu_0 H^2 - \varepsilon_0 E^2) \, dv \]

\[ \Delta f_{L,\text{stat}} = -K_L \cdot E_{acc}^2 \]

\( k_L \) – Lorentz coefficient

\( E \) and \( H \) at \( E_{acc} = 25 \text{ MV/m} \) in TESLA inner-cup
Thermal Breakdown

- Localized heating

- Thermal breakdown occurs when the heat generated at the hot spot is larger than that can be evacuated to the helium bath

- Both the thermal conductivity and the surface resistance of Nb are highly temperature dependent between 2 and 9K
Losses in RF Cavities
Losses in Normal Conducting Cavities

- Losses are given by Ohm’s Law $J = \sigma E$ where $\sigma$ is the conductivity

- In a cavity, rf magnetic field drives an oscillating current in the cavity wall

- Following Maxwell’s equations

\[ \nabla \times B = \mu j + \mu \varepsilon \frac{\partial E}{\partial t} \]
\[ \nabla \times E = -\frac{\partial B}{\partial t} \]

- Neglecting displacement current

\[ \nabla^2 B - i\mu \sigma \omega B = 0 \]
Losses in Normal Conducting Cavities

• Considering the cavity as wall as a local plane surface, solve one dimensional problem at the surface for uniform magnetic field in y direction

\[ \nabla^2 B - i \mu \sigma \omega B = 0 \quad \Rightarrow \quad H_y = H_0 e^{-\frac{1+i}{\delta} x} \]

• with field decaying into the conductor over the skin depth

\[ \delta = \left( \frac{2}{\mu_0 \omega \sigma} \right)^{1/2} \]
Losses in Normal Conducting Cavities

• From Maxwell equation

\[ E_z = \frac{1 + i}{\sigma \delta} H_y \]

- A small tangential electric field component decays into the conductor

• Surface impedance

\[ H_y(z) = \frac{(1 - i)}{\mu_0 \omega \delta} E_z(z) \]

\[ Z = \frac{E_z(0)}{H_y(0)} = \frac{(1 + i)}{2} \mu_0 \omega \delta = \frac{(1 + i)}{\sigma \delta} = (1 + i) \left( \frac{\mu_0 \omega}{2\sigma} \right)^{1/2} \]

• Surface resistance is the real part of surface impedance

\[ R_s = \frac{1}{\sigma \delta} = \sqrt{\frac{\mu_0 \omega}{2\sigma}} \]
Superconductivity

- **Superconductivity** – Discovered in 1911 by Kammerlingh-Onnes, is a phenomenon where below a certain temperature, called the critical temperature ($T_c$), some materials show a sudden drop of the dc electrical resistance to zero.

Kamerlingh Onnes and van der Waals in Leiden with the helium 'liquefactor' (1908)
Meissner Effect

Meissner Effect – Discovered by Meissner and Ochsenfeld in 1933
- Ability to completely expel an externally applied magnetic field when cooled down below the critical temperature ($T_c$)
- Superconductor behaves as a perfect diamagnet

- Surface currents created at the surface generates a magnetic field that cancels the external magnetic field inside the superconductor
- These surface currents do not decay with time due to the zero resistance in the superconductor
Critical Magnetic Field

- Critical magnetic field \( (B_c) \) – Field beyond at which superconductivity is destroyed

\[
B_c(T) = B_c(0) \left[ 1 - \left( \frac{T}{T_c} \right)^2 \right]
\]

- Type I – Soft Superconductors
- Type II – Hard Superconductors
Magnetization

- Meissner state in superconductor is an ideal diamagnet
- Magnetization: \( M = B - \mu_0 H \)

- Complete Meissner state - Type I superconductor
- Partial Meissner state – Type II superconductor
Theories of Superconductivity

- Gorter and Casimir (1934) – Two Fluid Model
    → London Penetration Depth (\(\lambda\))
  - Non local generalization to London equations by Pippard
    → Pippard Coherence Length (\(\xi\))

- Ginzburg Landau Theory (1950)
  - Second order phase transition of complex order parameter (\(\Psi\))

- BCS Theory (Bardeen Cooper Schrieffer) (1957)
  - Microscopic theory
  - Two fluid mode revised
Two Fluid Model

- Macroscopic theory of superconductivity
- Coexistence of:
  - super electrons \( (n_s) \)
  - normal electrons \( (n_n) \)
  - Total density \( n = n_s + n_n \)
- Only super electrons are accelerated by the constant electric field \( (E) \)
- Super current density \( \vec{J}_s = -en_s\vec{v} \)
- Yields First London Equation \( \frac{\partial \vec{J}_s}{\partial t} = \frac{n_s e^2}{m} \vec{E} \)
- Super electrons are not affected by the normal electrons \( \vec{J}_n = \sigma_n \vec{E} \)
London Equations

- Using Maxwell’s equations
  \[
  \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_s
  \]

- Yield Second London Equation
  \[
  \nabla^2 \vec{B} - \frac{\mu_0 n_s e^2}{m} \vec{B} = 0
  \]

- Field penetration in the superconductor
  \[
  B(x) = B(0) \exp \left[-\frac{x}{\lambda_L}\right]
  \]

- London penetration depth
  \[
  \lambda_L = \left[\frac{m}{\mu_0 n_s e^2}\right]^{1/2}
  \]
Fundamental Lengths

• London penetration depth ($\lambda_L$)
  – Distance over which magnetic fields decay in superconductors

• Pippard coherence length ($\xi_0$)
  – Distance over which the superconducting state decays

• Type II superconductors – $\lambda_L >> \xi_0$
• Type I superconductors – $\xi_0 >> \lambda_L$
Ginzburg Landau Theory

• Linear London equations

\[ \frac{\partial \bar{J}_s}{\partial t} = -\frac{\bar{E}}{\lambda^2 \mu_0} \]
\[ \nabla^2 \bar{H} - \frac{1}{\lambda^2} \bar{H} = 0 \]

• Describes the electrodynamics of superconductors at all \( T \) if:
  – Current density \( J_s \) is small
  – Density of super electrons \( (n_s) \) is spatially uniform

• Many important phenomena in superconductivity occur because \( (n_s) \) is not uniform
  – Interfaces between normal and superconductors
  – Trapped flux
  – Intermediate state

• G-L Theory – Nobel prize in 2003
Ginzburg Landau Theory

• G-L Theory – Generalization of London equations to nonlinear problems but still retain the local approximation of the electrodynamics

• Theory of second order phase transition is based on an order parameter which is zero above the transition temperature and non-zero below

• For superconductors, G-L theory uses a complex order parameter $\psi(r)$ such that $|\psi(r)|^2$ represents the density of super electrons

$$\psi (\vec{r}) = |\psi|exp (i \phi (\vec{r})) \quad \phi(r) \text{ is the phase}$$
Ginzburg Landau Theory

• London penetration depth:

\[ \lambda_L(T) = \left( \frac{m^* \beta}{2e^2 \alpha'} \right)^{1/2} \sqrt{\frac{T_c}{T_c - T}} \]

• Coherence length:

\[ \xi(T) = \left( \frac{\hbar^2}{4m^* \alpha'} \right)^{1/2} \sqrt{\frac{T_c}{T_c - T}} \]

• GL parameter: \( \kappa = \frac{\lambda(T)}{\xi(T)} \) is independent of \( T \)

• Critical field \( H_c(T) \)

\[ H_c(T) = \frac{\phi_0}{2\sqrt{2} \xi(T) \lambda_L(T)} \]
### Material Parameters

<table>
<thead>
<tr>
<th>Superconductor</th>
<th>$\lambda_L(0)$ (nm)</th>
<th>$\xi_0$ (nm)</th>
<th>$\kappa$</th>
<th>$2\Delta(0)/kT_c$</th>
<th>$T_c$(K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al</td>
<td>16</td>
<td>1500</td>
<td>0.011</td>
<td>3.40</td>
<td>1.18</td>
</tr>
<tr>
<td>In</td>
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<td>300</td>
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<td>Nb</td>
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<td>39</td>
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<td>8.95-9.2</td>
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<tr>
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<td>0.38</td>
<td>3.55</td>
<td>4.46</td>
</tr>
<tr>
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<td>6</td>
<td>8.3</td>
<td>4.4</td>
<td>18</td>
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<tr>
<td>NbN</td>
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<td>6</td>
<td>8.3</td>
<td>4.3</td>
<td>$\leq 17$</td>
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<tr>
<td>Y$_{Ba_2}Cu_3O_x$</td>
<td>140</td>
<td>1.5</td>
<td>93</td>
<td>4.5</td>
<td>90</td>
</tr>
</tbody>
</table>
BCS Theory

- Bardeen-Cooper-Schrieffer Theory (1957) – Nobel prize in 1972
- Macroscopical $\rightarrow$ Microscopical representation

Low-Temperature Superconductivity

December was the 50th anniversary of the theory of superconductivity, the flow of electricity without resistance that can occur in some metals and ceramics.

Electrons carrying an electrical current through a metal wire typically encounter resistance, which is caused by collisions and scattering as the particles move through the vibrating lattice of metal atoms.

Electrical Resistance

Critical Temperature

As the metal is cooled to low temperatures, the lattice vibration slows. A moving electron attracts nearby metal atoms, which create a positively charged wake behind the electron. This wake can attract another nearby electron.

Cooper Pairs

The two electrons form a weak bond, called a Cooper pair, which encounters less resistance than two electrons moving separately. When more Cooper pairs form, they behave in the same way.

Superconductivity

If a pair is scattered by an impurity, it will quickly get back in step with other pairs. This allows the electrons to flow undisturbed through the lattice of metal atoms. With no resistance, the current may persist for years.

Sources: Oak Ridge National Laboratory; Philip W. Phillips

Jonathan Corum/The New York Times
Cooper Pairs

- Cooper pairs – Pair of electrons formed due to electron-phonon interaction that dominates over the repulsive Coulomb force

- Moving electron distorts the lattice and leaves behind a trail of positive charge that attracts another electron moving in opposite direction

- Has lower energy than the two separate electrons

- Therefore, electron pairs form bound states of lower energy which are stable than the Fermi ground state

- Strong overlap of many Cooper pairs results in the macroscopic phase coherence
Energy Gap

- Energy gap ($\Delta$) – gap around Fermi level between ground state and excited state.

- At $0 < T < T_c$ not all electrons are bounded in to Cooper pairs.

- Density of unpaired electrons is given by:

$$n_{\text{normal}} \propto \exp\left(-\frac{\Delta}{k_B T}\right)$$
Losses in Superconductor

• At the presence of an rf field
  – Cooper pairs move without resistance \( \Rightarrow \) Do not dissipate power
  – Due to inertial mass of Cooper pairs they cannot follow an AC electromagnetic fields instantly and do not shield it perfectly
  – Remaining residual field accelerates the normal electrons that dissipate power
• More normal electrons \( \Rightarrow \) The material is more lossy
  – Losses decrease with temperature below \( T_c \)
• Faster the field oscillates the less perfect the shielding
  – Losses increase with frequency
Surface Impedance

- Following two fluid model
  
  \[ \tilde{J}_s = \frac{n_e e^2}{m \omega} \tilde{E} \quad \tilde{J}_n = -\frac{i}{\mu_0 m \lambda_L^2} \tilde{E} \]

- Total current density
  
  \[ \tilde{J} = \tilde{J}_n + \tilde{J}_s = (\sigma_n - i \sigma_s) \tilde{E} \]

- Surface Impedance:
  
  \[ Z_s = \sqrt{\frac{\omega \mu_0}{\sigma}} (1 + i) \]

- Skin depth
  
  \[ \delta = \sqrt{\frac{2}{\omega \mu_0 \sigma}} \approx (1 + i) \lambda_L \left( 1 + i \frac{\sigma_n}{2 \sigma_s} \right) \]

  \[ H_y = H_0 e^{-x/\lambda_L} e^{-ix\sigma_n/2\sigma_s^2} \]
**BCS Surface Resistance**

- Surface impedance
  
  \[ Z_s \approx \sqrt{\frac{\omega \mu_0}{\sigma_s}} \left( \frac{\sigma_n}{2\sigma_s} + i \right) = R_s + iX_s \]

  \[ X_s = \omega \mu_0 \lambda_L \quad R_s = \frac{1}{2} \approx \sigma_n \omega^2 \mu_0^2 \lambda_L^3 \]

- Taking may parameters into account Mattis and Bardeen developed theory based on BCS theory

  \[ R_{BCS} = A \frac{\omega^T}{T} \exp \left( \frac{-\Delta}{k_B T} \right) \]

  \[ A \text{ – material parameters} \]
Surface Resistance

- Surface resistance of superconductors ($R_s$)
  \[ R_s = R_{BCS} + R_{res} \quad [\Omega] \]

- Residual resistance ($R_{res}$) due to:
  - Dielectric surface contaminants (gases, chemical residues, ..)
  - Normal conducting defects, inclusions
  - Surface imperfections (cracks, scratches, ..)
  - Trapped flux during cool down through critical temperature
  - Hydrogen absorption during chemical processing

- An approximation of $R_{BCS}$ for Nb:
  \[ R_{BCS} \approx 2 \times 10^{-4} \left( \frac{f \text{ [GHz]}}{1.5} \right)^2 \frac{1}{T} \exp\left( -\frac{17.67}{T} \right) \quad [\Omega] \]