Betatron Motion with Coupling of Horizontal and Vertical Degrees of Freedom – Part I

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Outline

- Introduction
- Equations of Motion, Symplecticity and Eigen-vectors
- Eigen-vectors and Particle Ellipsoid in 4D Phase Space
- Generalized Twiss Functions
- Derivatives of Tunes and Beta-Functions – 4D Floque formulae
- Second order moments in terms of generalized Twiss functions
  - V. Lebedev, A. Bogacz, 'Betatron Motion with Coupling of Horizontal and Vertical Degrees of Freedom', 2000, http://dx.doi.org/10.1088/1748-0221/5/10/P10010

Introduction

- Courant-Snyder representation for one-dimensional betatron motion
  - Simple relations between Twiss parameters, eigen-vectors and bilinear form for the particle ellipsoid
  - Symplecticity $\Rightarrow$ $2\times2 - 1 = 3$ parameters
- From uncoupled to strongly coupled motion by design
  - "Moebius Twist Accelerator" to create round beams (Cornell)
  - Ionization cooling channel for Neutrino Factory and Muon Collider
  - Vertex to plane adapter for electron cooling (Fermilab)

Two dimensional coupled betatron motion

- Symplecticity $\Rightarrow$ $4\times4 - 6 = 10$ parameters
  - Effective parameterization in terms of generalized Twiss functions
- Shortcomings of the existing representations
  - Edwards and Teng, Fermilab (1973)
    - Ambiguity of the rotation angle
    - Physical meaning of the betatron phase advance?
  - G. Ripken, et al., DESY (1987)
    - Oriented for circular accelerators
    - Incomplete parametrization (one needs 10 independent parameters to fully describe 2D betatron motion)

Unresolved issues for both parametrizations

- Quest for versatile representation conveniently describing both storage rings and transfer lines
- 2D emittances - how are they related to the 4D beam emittance?
- How to determine the beam emittances and the generalized Twiss parameters from the particle beam ellipsoid (bilinear form), and from the second-order moments of the particle distribution?

Equations of Motion and Symplecticity Condition

- Two-dimensional linear motion

$$x'' + \left(K_x^{\ 2} + k\right)x + \left(N - \frac{1}{2}R'\right)y - Ry' = 0 \quad ,$$

$$y'' + \left(K_y^{\ 2} - k\right)y + \left(N + \frac{1}{2}R'\right)x + Rx' = 0 \quad .$$

$K_{x,y} = eB_{y,x} / Pc$ - dipole

$k = eG / Pc$ - quadrupole

$N = eG_s / Pc$ - skew-quadrupole

$R = eB_s / Pc$ - longitudinal magnetic field

Hamiltonian formulation - equations of motion

$$\frac{d\hat{\mathbf{x}}}{ds} = \mathbf{UH}\hat{\mathbf{x}}$$

- Hamiltonian matrix:

$$\mathbf{H} = \begin{bmatrix} K_x^{\ 2} + k + \frac{R^2}{4} & 0 & N & -R/2 \\ 0 & 1 & R/2 & 0 \\ N & R/2 & K_y^{\ 2} - k + \frac{R^2}{4} & 0 \\ -R/2 & 0 & 0 & 1 \end{bmatrix}$$

- Unit symplectic matrix :

$$\mathbf{U} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\mathbf{U}^T = -\mathbf{U}$$

$$\mathbf{U}\mathbf{U} = -\mathbf{I}$$

$$\mathbf{U}\mathbf{U}^T = \mathbf{I}$$

Hamiltonian formulation - equations of motion

- Canonical variables

$$p_x = x' - \frac{R}{2} y,$$
$$p_y = y' + \frac{R}{2} x.$$

$R = eB_s / Pc$ - longitudinal magnetic field

- Relation between geometrical and canonical variables

$$\hat{\mathbf{x}} = \mathbf{R}\mathbf{x} \quad ,$$

where

$$\hat{\mathbf{x}} \equiv \begin{bmatrix} x \\ p_x \\ y \\ p_y \end{bmatrix} \quad , \quad \mathbf{x} \equiv \begin{bmatrix} x \\ \theta_x \\ y \\ \theta_y \end{bmatrix} \quad , \quad \mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -R/2 & 0 \\ 0 & 0 & 1 & 0 \\ R/2 & 0 & 0 & 1 \end{bmatrix} \quad ,$$

A ‘cap’ denotes transfer matrices and vectors related to the canonical variables.

Hamiltonian formulation - equations of motion

- Lagrange invariant

$$\frac{d}{ds}\left(\hat{\mathbf{x}}_1^{\ T}\mathbf{U}\hat{\mathbf{x}}_2\right)=\frac{d\hat{\mathbf{x}}_1^{\ T}}{ds}\mathbf{U}\hat{\mathbf{x}}_2+\hat{\mathbf{x}}_1^{\ T}\mathbf{U}\frac{d\hat{\mathbf{x}}_2}{ds}=\hat{\mathbf{x}}_1^{\ T}\mathbf{H}^T\mathbf{U}^T\mathbf{U}\hat{\mathbf{x}}_2+\hat{\mathbf{x}}_1^{\ T}\mathbf{U}\mathbf{U}\mathbf{H}\hat{\mathbf{x}}_2=0 \quad ,$$

$$\boxed{\hat{\mathbf{x}}_1^{\ T}\mathbf{U}\hat{\mathbf{x}}_2=\mathrm{inv}}$$

- Transfer matrix for canonical variables

$$\hat{\mathbf{x}}=\hat{\mathbf{M}}(0,s)\hat{\mathbf{x}}_0$$

- Symplecticity condition

$$\hat{\mathbf{x}}_0^{\ T}\mathbf{U}\hat{\mathbf{x}}_0=\hat{\mathbf{x}}_0^{\ T}\hat{\mathbf{M}}(0,s)^T\mathbf{U}\hat{\mathbf{M}}(0,s)\hat{\mathbf{x}}_0=\mathrm{inv}$$

- The above equation is satisfied for any $\hat{\mathbf{x}}$

Hamiltonian formulation - Sympecticity

$$\boxed{\hat{\mathbf{M}}(0,s)^T\,\mathbf{U}\,\hat{\mathbf{M}}(0,s) = \mathbf{U}}$$

- Six independent equations – matrix $\hat{\mathbf{M}}(0,s)^T\,\mathbf{U}\,\hat{\mathbf{M}}(0,s)$ is antisymmetric $\Rightarrow$ only 10 out of 16 elements of the transfer matrix are independent

Eigen-vectors

$$\hat{\mathbf{M}}\hat{\mathbf{v}}_i = \lambda_i \hat{\mathbf{v}}_i \quad , \qquad i = 1, 2, 3, 4$$

- For any two eigen-vectors the symplecticity condition yields

$$0 = \lambda_j \hat{\mathbf{v}}_j^{\,T} \mathbf{U}\left(\hat{\mathbf{M}}\hat{\mathbf{v}}_i - \lambda_i \hat{\mathbf{v}}_i\right) = \left(\hat{\mathbf{M}}\hat{\mathbf{v}}_j\right)^T \mathbf{U}\hat{\mathbf{M}}\hat{\mathbf{v}}_i - \lambda_j \hat{\mathbf{v}}_j^{\,T} \mathbf{U}\lambda_i \hat{\mathbf{v}}_i = \left(1 - \lambda_j \lambda_i\right)\hat{\mathbf{v}}_j^{\,T} \mathbf{U}\hat{\mathbf{v}}_i$$

- The eigen-values always appear in two reciprocal pairs
  - For stable betatron motion
    - $|\lambda_i| = 1$ and $\lambda_i \neq \pm 1$
    - the four eigen-values split into two complex conjugate pairs: λ_l, λ_l^*, l =1, 2
- Four eigen-vectors – two complex conjugate pairs: $\hat{\mathbf{v}}_l, \hat{\mathbf{v}}_l^*$, l = 1, 2.

Eigen-vectors

- Orthogonality conditions:

$$\hat{\mathbf{v}}_1^{+}\mathbf{U}\hat{\mathbf{v}}_1 \neq 0 \,,$$
$$\hat{\mathbf{v}}_2^{+}\mathbf{U}\hat{\mathbf{v}}_2 \neq 0 \,,$$
$$\hat{\mathbf{v}}_i^{T}\mathbf{U}\hat{\mathbf{v}}_j = 0 \,, \qquad \text{if } i \neq j,$$

  - Top two expressions are purely imaginary

$$\left(\hat{\mathbf{v}}_l^{+}\mathbf{U}\hat{\mathbf{v}}_l\right)^{*} = \left(\hat{\mathbf{v}}_l^{+}\mathbf{U}\hat{\mathbf{v}}_l\right)^{+} = \hat{\mathbf{v}}_l^{+}\mathbf{U}^{+}\hat{\mathbf{v}}_l = -\hat{\mathbf{v}}_l^{+}\mathbf{U}\hat{\mathbf{v}}_l \quad , \quad l = 1,\, 2.$$

Eigen-vectors

- Eigen-vector normalization

$$\hat{\mathbf{v}}_1^{\,+}\mathbf{U}\hat{\mathbf{v}}_1 = -2i \quad , \quad \hat{\mathbf{v}}_2^{\,+}\mathbf{U}\hat{\mathbf{v}}_2 = -2i \quad ,$$
$$\hat{\mathbf{v}}_1^{\,T}\mathbf{U}\hat{\mathbf{v}}_1 = 0 \quad , \quad \hat{\mathbf{v}}_2^{\,T}\mathbf{U}\hat{\mathbf{v}}_2 = 0 \quad ,$$
$$\hat{\mathbf{v}}_2^{\,T}\mathbf{U}\hat{\mathbf{v}}_1 = 0 \quad , \quad \hat{\mathbf{v}}_2^{\,+}\mathbf{U}\hat{\mathbf{v}}_1 = 0 \quad .$$

- $2\times4\times2 - 6 = 10$ (8 scalars and 2 initial phases to parameterize eigen-vectors)

Eigen-vectors and Particle Ellipsoid in 4D Space

❖Particle position/angle vector at the beginning of the lattice

$$\hat{\mathbf{x}} = \mathrm{Re}\left(A_1 e^{-i\psi_1}\hat{\mathbf{v}}_1 + A_2 e^{-i\psi_2}\hat{\mathbf{v}}_2\right)$$

where, A_1, A_2, ψ_1 and ψ_2 , are the betatron amplitudes and phases.

❖Let us introduce the following real matrix:

$$\hat{\mathbf{V}} = \left[\hat{\mathbf{v}}_1{}', -\hat{\mathbf{v}}_1{}'', \hat{\mathbf{v}}_2{}', -\hat{\mathbf{v}}_2{}''\right] \quad .$$

- $\hat{\mathbf{V}}$ is a symplectic matrix (a direct consequence of eigen-vector orthogonality):
  - $\hat{\mathbf{V}}^T \mathbf{U} \hat{\mathbf{V}} = \mathbf{U}$

Eigen-vectors and Particle Ellipsoid in 4D Space

$$\hat{\mathbf{V}}^T \mathbf{U} \hat{\mathbf{V}} = \mathbf{U}$$

- matrix $\hat{\mathbf{V}}$ symplecticity yields a useful identity for the inverse of $\hat{\mathbf{V}}$:
  - $\hat{\mathbf{V}}^{-1} = -\mathbf{U}\hat{\mathbf{V}}^T\mathbf{U}$

- Multi-particle beam emittance - an ensemble of particles, whose motion is confined to a 4D ellipsoid. A 3D surface of this ellipsoid, determined by particles with extreme betatron amplitudes can be described in terms of a bilinear form

$$\hat{\mathbf{x}}^T \hat{\mathbf{\Xi}} \hat{\mathbf{x}} = 1 \quad .$$

Eigen-vectors and Particle Ellipsoid in 4D Space

- Using matrix $\hat{\mathbf{V}}$ one can express a position/angle vector as follows:

$$\hat{\mathbf{x}} = \hat{\mathbf{V}}\mathbf{A}\boldsymbol{\xi}$$

where

$$\mathbf{A} = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & A_2 & 0 \\ 0 & 0 & 0 & A_2 \end{bmatrix} , \quad \boldsymbol{\xi} = \begin{bmatrix} \cos\psi_1 \cos\psi_3 \\ -\sin\psi_1 \cos\psi_3 \\ \cos\psi_2 \sin\psi_3 \\ -\sin\psi_2 \sin\psi_3 \end{bmatrix} .$$

- the third parameter ψ_3 is introduced, so that the vector $\boldsymbol{\xi}$ would describe a 3D sphere with a unit radius

$$\boldsymbol{\xi}^T\boldsymbol{\xi} = 1 \quad , \quad \boldsymbol{\xi} = \left(\hat{\mathbf{V}}\mathbf{A}\right)^{-1}\hat{\mathbf{x}}$$

$$\hat{\mathbf{x}}^T\left(\left(\hat{\mathbf{V}}\mathbf{A}\right)^{-1}\right)^T\left(\hat{\mathbf{V}}\mathbf{A}\right)^{-1}\hat{\mathbf{x}} = 1 \qquad \Rightarrow \qquad \hat{\boldsymbol{\Xi}} = \mathbf{U}\hat{\mathbf{V}}\mathbf{A}^{-1}\mathbf{A}^{-1}\hat{\mathbf{V}}^T\mathbf{U}^T .$$

Beam emittance 4-D

- Matrix $\hat{\boldsymbol{\Xi}}$ can be diagonalized as follows

$$\hat{\mathbf{V}}^T \hat{\boldsymbol{\Xi}} \hat{\mathbf{V}} = \mathbf{A}^{-1}\mathbf{A}^{-1} \equiv \hat{\boldsymbol{\Xi}}' \quad .$$

- The symplectic transform $\hat{\mathbf{V}}$
  - reduces matrix $\hat{\boldsymbol{\Xi}}$ to its diagonal form
  - 4D volume of the ellipsoid remains unchanged, since $\det \hat{\mathbf{V}} = 1$
- In the new coordinates particle beam ellipsoid can be written as:

$$\hat{\Xi}'_{11} x'^2 + \hat{\Xi}'_{22} p_x'^2 + \hat{\Xi}'_{33} y'^2 + \hat{\Xi}'_{44} p_y'^2 = 1$$

Beam emittance 4-D

- 4D beam emittance (ellipsoid volume) can be expressed as follows:

$$\varepsilon_{4D} = \frac{1}{\sqrt{\hat{\Xi}'_{11}\hat{\Xi}'_{22}\hat{\Xi}'_{33}\hat{\Xi}'_{44}}} = \frac{1}{\sqrt{\det(\hat{\Xi}')}} = \frac{1}{\sqrt{\det(\hat{\Xi})}} = (A_1 A_2)^2$$

$$\varepsilon_{4D} = \varepsilon_1 \varepsilon_2 = \frac{1}{\sqrt{\det(\hat{\Xi})}} \quad , \quad \varepsilon_1 = A_1^{\ 2} \quad , \quad \varepsilon_2 = A_2^{\ 2}$$

Beam emittance 4-D

- Knowing beam emittances and the eigen-vectors (matrix $\hat{\mathbf{V}}$), the beam ellipsoid can be described in the following compact form

$$\hat{\mathbf{x}}^T \hat{\boldsymbol{\Xi}} \hat{\mathbf{x}} = 1$$

$$\hat{\boldsymbol{\Xi}} = \mathbf{U}\hat{\mathbf{V}} \begin{bmatrix} 1/\varepsilon_1 & 0 & 0 & 0 \\ 0 & 1/\varepsilon_1 & 0 & 0 \\ 0 & 0 & 1/\varepsilon_2 & 0 \\ 0 & 0 & 0 & 1/\varepsilon_2 \end{bmatrix} \hat{\mathbf{V}}^T \mathbf{U}^T$$

Second order moments of the particle distribution

- Gaussian distribution for 2D coupled betatron motion

$$f(\hat{\mathbf{x}}) = \frac{1}{4\pi^2 \varepsilon_1 \varepsilon_2} \exp\left(-\frac{1}{2} \hat{\mathbf{x}}^T \hat{\boldsymbol{\Xi}} \hat{\mathbf{x}} \right)$$

- Second order moments of the distribution

$$\hat{X}_{ij} \equiv \overline{\hat{x}_i \hat{x}_j} = \int \hat{x}_i \hat{x}_j f(\hat{\mathbf{x}}) d\hat{x}^4 = \frac{1}{4\pi^2 \varepsilon_1 \varepsilon_2} \int \hat{x}_i \hat{x}_j \exp\left(-\frac{1}{2} \hat{\mathbf{x}}^T \hat{\boldsymbol{\Xi}} \hat{\mathbf{x}} \right) d\hat{x}^4$$

Beam emittance 4-D

- Applying coordinate transformation, $\hat{\mathbf{y}} = \hat{\mathbf{V}}^{-1}\hat{\mathbf{x}}$, (matrix $\hat{\mathbf{\Xi}}$ is reduced to its diagonal form) makes the above integration trivial. The final result is :

$$\hat{\mathbf{X}} = \hat{\mathbf{V}} \begin{bmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & \varepsilon_1 & 0 & 0 \\ 0 & 0 & \varepsilon_2 & 0 \\ 0 & 0 & 0 & \varepsilon_2 \end{bmatrix} \hat{\mathbf{V}}^T$$

- One can prove by direct substitution that

$$\hat{\mathbf{X}} = \hat{\mathbf{\Xi}}^{-1} \quad .$$

Beam emittance 4-D

❖ How to find the beam emittances and the eigen-vectors if one knows $\hat{\mathbf{X}}$ or $\hat{\mathbf{\Xi}}$?

- The following characteristic equation:

$$\det\left(\hat{\mathbf{\Xi}} - i\lambda\,\mathbf{U}\right) = 0$$

has 4 roots: $\lambda_1 = -\lambda_2 = 1/\varepsilon_1$ and $\lambda_3 = -\lambda_4 = 1/\varepsilon_2$

♠ Proof:

$$\det\left(\hat{\mathbf{\Xi}} - i\lambda\,\mathbf{U}\right) = \det\left(\mathbf{U}\hat{\mathbf{V}}\hat{\mathbf{\Xi}}'\hat{\mathbf{V}}^T\mathbf{U}^T - i\lambda\,\mathbf{U}\right) = \det\left(\hat{\mathbf{\Xi}}' - i\lambda\,\mathbf{U}^T\hat{\mathbf{V}}^T\mathbf{U}\hat{\mathbf{V}}\mathbf{U}\right) =$$

$$\det\left(\hat{\mathbf{\Xi}}' - i\lambda\,\mathbf{U}\right) = \left(\frac{1}{{\varepsilon_1}^2} - \lambda^2\right)\left(\frac{1}{{\varepsilon_2}^2} - \lambda^2\right) = 0 \quad .$$

Beam emittance 4-D

- Then, the eigen-vectors are determined by solving the following equation:

$$\left(\hat{\boldsymbol{\Xi}} - \frac{i}{\varepsilon_l}\mathbf{U}\right)\hat{\mathbf{v}}_l = 0$$

  - Proof:
    - Rewrite equation, $\hat{\boldsymbol{\Xi}} = \mathbf{U}\hat{\mathbf{V}}\hat{\boldsymbol{\Xi}}'\hat{\mathbf{V}}^T\mathbf{U}^T$. as $\boxed{\hat{\boldsymbol{\Xi}}\hat{\mathbf{V}}\mathbf{U} - \mathbf{U}\hat{\mathbf{V}}\hat{\boldsymbol{\Xi}}' = 0}$
    - multiply both sides of the above equation by vectors $\mathbf{u}_l$, l = 1, 2

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -i \\ 0 \\ 0 \end{bmatrix} \quad , \qquad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -i \end{bmatrix}$$

Beam emittance 4-D

- and employing the following properties of the vectors $\mathbf{u}_l$, l = 1, 2:

$$\hat{\mathbf{V}}\mathbf{u}_l = \hat{\mathbf{v}}_l , \qquad \mathbf{U}\mathbf{u}_l = -i\mathbf{u}_l \quad \text{and} \quad \mathbf{\Xi}'\mathbf{u}_l = \frac{1}{\varepsilon_l}\mathbf{u}_l .$$

- one obtains the desired equation: $\left(\hat{\mathbf{\Xi}} - \frac{i}{\varepsilon_l}\mathbf{U}\right)\hat{\mathbf{v}}_l = 0$, l = 1, 2

♦ Similar equation holds for the second order moments

$$\det\left(\hat{\mathbf{X}}\mathbf{U} + i\lambda\,\mathbf{I}\right) = 0 . \qquad \varepsilon_l = \lambda_l \quad , \; l = 1, 2$$

and

$$\left(\hat{\mathbf{X}}\mathbf{U} + i\varepsilon_l\mathbf{I}\right)\hat{\mathbf{v}}_l = 0 \quad , \; l = 1, 2$$

♦ That yields another useful way of expressing the 4D emittance

$$\varepsilon_{4D} = \varepsilon_1\varepsilon_2 = \sqrt{\det\left(\hat{\mathbf{X}}\right)} \quad .$$

Twiss Functions for Coupled 2D Motion

❖ Single-particle phase-space trajectory along the beam orbit

$$\hat{\mathbf{x}}(s) = \hat{\mathbf{M}}(0,s)\,\mathrm{Re}\left(\sqrt{\varepsilon_1}\,\hat{\mathbf{v}}_1 e^{-i\psi_1} + \sqrt{\varepsilon_2}\,\hat{\mathbf{v}}_2 e^{-i\psi_2}\right)$$

$$= \mathrm{Re}\left(\sqrt{\varepsilon_1}\,\hat{\mathbf{v}}_1(s) e^{-i(\psi_1+\mu_1(s))} + \sqrt{\varepsilon_2}\,\hat{\mathbf{v}}_2(s) e^{-i(\psi_2+\mu_2(s))}\right) \quad ,$$

- vectors $\hat{\mathbf{v}}_1(s)$ and $\hat{\mathbf{v}}_2(s)$ are the eigen-vectors at coordinate *s*
- ψ_1 and ψ_2 are the initial phases of betatron motion
- The phase terms $e^{-i\mu_1(s)}$ and $e^{-i\mu_2(s)}$ are introduced to put the eigen-vectors into the following standard form:

Twiss Functions for Coupled 2D Motion

$$\hat{\mathbf{v}}_1(s) = \begin{bmatrix} \sqrt{\beta_{1x}(s)} \\ -\dfrac{iu_1(s)+\alpha_{1x}(s)}{\sqrt{\beta_{1x}(s)}} \\ \sqrt{\beta_{1y}(s)}e^{i\nu_1(s)} \\ -\dfrac{iu_2(s)+\alpha_{1y}(s)}{\sqrt{\beta_{1y}(s)}}e^{i\nu_1(s)} \end{bmatrix}, \quad \hat{\mathbf{v}}_2(s) = \begin{bmatrix} \sqrt{\beta_{2x}(s)}e^{i\nu_2(s)} \\ -\dfrac{iu_3(s)+\alpha_{2x}(s)}{\sqrt{\beta_{2x}(s)}}e^{i\nu_2(s)} \\ \sqrt{\beta_{2y}(s)} \\ -\dfrac{iu_4(s)+\alpha_{2y}(s)}{\sqrt{\beta_{2y}(s)}} \end{bmatrix},$$

- $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ are selected out of two complex conjugate pairs, so that u_1 ,u_4 >0

❖Generalized Twiss functions (10 independent parameters):

- $\mu_1(s)$ and $\mu_2(s)$ are the phase advances of betatron motion.
- $\beta_{1x}(s)$, $\beta_{1y}(s)$, $\beta_{2x}(s)$ and $\beta_{2y}(s)$ are the beta-functions;
- $\alpha_{1x}(s)$, $\alpha_{1y}(s)$, $\alpha_{2x}(s)$ and $\alpha_{2y}(s)$ are the alpha-functions

Twiss Functions for Coupled 2D Motion

- Introduced six real functions $u_1(s)$, $u_2(s)$, $u_3(s)$, $u_4(s)$, $\nu_1(s)$ and $\nu_2(s)$ are determined from the symplecticity condition

- The first three conditions yield:

$$u_1 = 1 - u_2 \,, \qquad u_4 = 1 - u_3 \quad \text{and} \qquad u_2 = u_3$$

- Then, one obtains

$$\hat{\mathbf{v}}_1 = \begin{bmatrix} \sqrt{\beta_{1x}} \\ -\dfrac{i(1-u)+\alpha_{1x}}{\sqrt{\beta_{1x}}} \\ \sqrt{\beta_{1y}}e^{i\nu_1} \\ -\dfrac{iu+\alpha_{1y}}{\sqrt{\beta_{1y}}}e^{i\nu_1} \end{bmatrix} \,, \qquad \hat{\mathbf{v}}_2 = \begin{bmatrix} \sqrt{\beta_{2x}}e^{i\nu_2} \\ -\dfrac{iu+\alpha_{2x}}{\sqrt{\beta_{2x}})}e^{i\nu_2} \\ \sqrt{\beta_{2y}} \\ -\dfrac{i(1-u)+\alpha_{2y}}{\sqrt{\beta_{2y}}} \end{bmatrix}$$

- For the uncoupled motion:

$$u = 0 \,, \quad \beta_{1y} = \beta_{2x} = 0 \qquad \text{and} \qquad \alpha_{1y} = \alpha_{2x} = 0$$

Twiss Functions for Coupled 2D Motion

❖ Explicit solution for $u(s)$

$$u = \frac{-\kappa_x^2\kappa_y^2 \pm \sqrt{\kappa_x^2\kappa_y^2\left(1+\frac{A_x^2-A_y^2}{\kappa_x^2-\kappa_y^2}\left(1-\kappa_x^2\kappa_y^2\right)\right)}}{1-\kappa_x^2\kappa_y^2}$$

where

$$A_x = \kappa_x\alpha_{1x} - \kappa_x^{-1}\alpha_{2x} \ ,$$

$$A_y = \kappa_y\alpha_{2y} - \kappa_y^{-1}\alpha_{1y} \ ,$$

$$\kappa_x = \sqrt{\frac{\beta_{2x}}{\beta_{1x}}}, \quad \kappa_y = \sqrt{\frac{\beta_{1y}}{\beta_{2y}}} \ .$$

- Time invariance (a positive displacement for a positive velocity)

Requires, $u \geq 0$ and $(1 - u) \geq 0 \quad \Rightarrow \quad 0 < u < 1$.

Twiss Functions for Coupled 2D Motion

❖General solution for $\nu_1(s)$ and $\nu_2(s)$

- Starting from the following expressions:

$$e^{i\nu_+} \equiv e^{i(\nu_2+\nu_2)} = \frac{A_x + i\left(\kappa_x(1-u) + \kappa_x^{-1}u\right)}{A_y - i\left(\kappa_y(1-u) - \kappa_y^{-1}u\right)} \quad ,$$

$$e^{i\nu_-} \equiv e^{i(\nu_2-\nu_2)} = \frac{A_x + i\left(\kappa_x(1-u) - \kappa_x^{-1}u\right)}{A_y + i\left(\kappa_y(1-u) - \kappa_y^{-1}u\right)} \quad ,$$

- one can get explicit solutions for ν_1 and ν_2:

$$\nu_1 = n\pi + \frac{1}{2}(\nu_+ - \nu_-) \quad ,$$

$$\nu_2 = m\pi + \frac{1}{2}(\nu_+ + \nu_-) \quad .$$

Twiss Functions for Coupled 2D Motion

$$\nu_1 = n\pi + \frac{1}{2}(\nu_+ - \nu_-) \quad ,$$

$$\nu_2 = m\pi + \frac{1}{2}(\nu_+ + \nu_-) \quad .$$

- ν_- and ν_+ are determined modulo 2π,
- which yields that ν_1 and ν_2 are determined modulo π.
- The last feature is a consequence of the fact that the mirror reflection does not affect β's and α's itself, but it changes relative signs of *x* and *y* components of the eigen-vectors (change of ν_1 and ν_2 by π).

Twiss Functions for Coupled 2D Motion

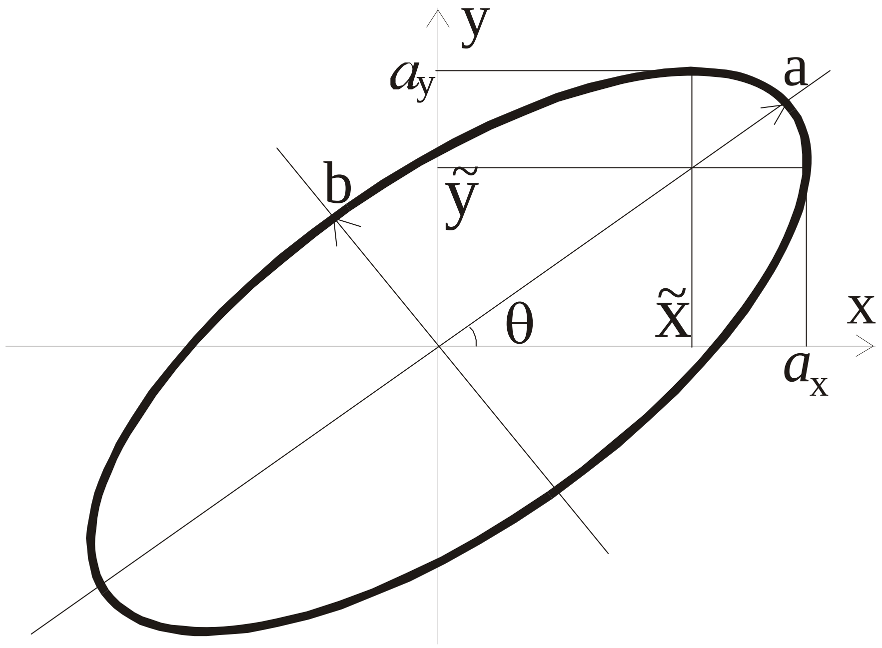
❖ Choice of eigen-vectors

- Weak coupling
  - $\hat{\mathbf{v}}_1$ – relates mostly to the horizontal motion
  - $\hat{\mathbf{v}}_2$ – relates mostly to the vertical motion.
- Strong coupling – the choice is arbitrary.
  - if one swaps two eigen-vectors it causes the following re-definitions:
    - $\beta_{1x} \leftrightarrow \beta_{2x}\,, \quad \beta_{1y} \leftrightarrow \beta_{2y}$
    - $\alpha_{1x} \leftrightarrow \alpha_{2x}\,, \quad \alpha_{1y} \leftrightarrow \alpha_{2y}$
    - $\nu_1 \to -\nu_2\,, \quad \nu_2 \to -\nu_1 \quad \text{and} \quad u \to 1-u\,.$

Beam sizes

$$a_x = \sqrt{\varepsilon_1 \beta_{1x} + \varepsilon_2 \beta_{2x}}$$

$$a_y = \sqrt{\varepsilon_1 \beta_{1y} + \varepsilon_2 \beta_{2y}}$$



- Ellipse equation

$$\frac{x^2}{a_x^2} - \frac{2\tilde{\alpha}xy}{a_x a_y} + \frac{y^2}{a_y^2} = 1 - \tilde{\alpha}^2$$

- Ellipse rotation parameter

$$\tilde{\alpha} \equiv \frac{\langle xy \rangle}{\sqrt{\langle x^2 \rangle \langle y^2 \rangle}} = \frac{\tilde{y}}{a_y} = \frac{\tilde{x}}{a_x} = \frac{\sqrt{\beta_{1x}\beta_{1y}}\varepsilon_1 \cos\nu_1 + \sqrt{\beta_{2x}\beta_{2y}}\varepsilon_2 \cos\nu_2}{\sqrt{\varepsilon_1 \beta_{1x} + \varepsilon_2 \beta_{2x}}\sqrt{\varepsilon_1 \beta_{1y} + \varepsilon_2 \beta_{2y}}}$$

Derivatives of Tunes and Beta-Functions

- A differential trajectory displacement related to the first eigen-vector

$$x(s+ds) = x(s) + x'(s)ds = x(s) + \left(p_x(s) + \frac{R}{2} y \right) ds =$$

$$\sqrt{\varepsilon_1}\,\mathrm{Re}\left(\left(\sqrt{\beta_{1x}(s)} + \left[-\frac{i(1-u(s)) + \alpha_{1x}(s)}{\sqrt{\beta_{1x}(s)}} + \frac{R}{2}\sqrt{\beta_{1y}(s)}\; e^{i\nu_1(s)} \right] ds \right) e^{-i(\mu_1(s)+\psi_1)} \right) \quad .$$

- Alternatively, the particle position can be expressed through the beta-functions at the new coordinate *s* + *ds*:

$$x(s+ds) = \mathrm{Re}\left(\sqrt{\varepsilon_1 \beta_x(s+ds)}\, e^{-i(\mu_1(s+ds)+\psi)} \right) =$$

$$\sqrt{\varepsilon_1}\,\mathrm{Re}\left(\left(\sqrt{\beta_{1x}(s)} + \frac{d\beta_{1x}}{2\sqrt{\beta_{1x}(s)}} - i\sqrt{\beta_{1x}(s)}\,d\mu \right) e^{-i(\mu_1(s)+\psi)} \right) \quad .$$

Derivatives of Tunes and Beta-Functions

- For the first eigen-vector

$$\frac{d\beta_{1x}}{ds} = -2\alpha_{1x} + R\sqrt{\beta_{1x}\beta_{1y}}\cos\nu_1 \quad , \qquad \frac{d\beta_{1y}}{ds} = -2\alpha_{1y} - R\sqrt{\beta_{1x}\beta_{1y}}\cos\nu_1 \quad ,$$

$$\frac{d\mu_1}{ds} = \frac{1-u}{\beta_{1x}} - \frac{R}{2}\sqrt{\frac{\beta_{1y}}{\beta_{1x}}}\sin\nu_1 \quad , \qquad \frac{d\mu_1}{ds} - \frac{d\nu_1}{ds} = \frac{u}{\beta_{1y}} + \frac{R}{2}\sqrt{\frac{\beta_{1x}}{\beta_{1y}}}\sin\nu_1 \quad ,$$

- For the second eigen-vector

$$\frac{d\beta_{2y}}{ds} = -2\alpha_{2y} - R\sqrt{\beta_{2x}\beta_{2y}}\cos\nu_2 \quad , \qquad \frac{d\beta_{2x}}{ds} = -2\alpha_{2x} + R\sqrt{\beta_{2x}\beta_{2y}}\cos\nu_2 \quad ,$$

$$\frac{d\mu_2}{ds} = \frac{1-u}{\beta_{2y}} + \frac{R}{2}\sqrt{\frac{\beta_{2x}}{\beta_{2y}}}\sin\nu_2 \quad , \qquad \frac{d\mu_2}{ds} - \frac{d\nu_2}{ds} = \frac{u}{\beta_{2x}} - \frac{R}{2}\sqrt{\frac{\beta_{2y}}{\beta_{2x}}}\sin\nu_2 \quad .$$

Transfer Matrix in terms of Twiss Functions

- Using the definition of the eigen-vectors one can derive the following identity

$$\hat{\mathbf{M}}\,\hat{\mathbf{V}} = \hat{\mathbf{V}}\,\mathbf{S} \quad ,$$

where the matrix **S** is defined as:

$$\mathbf{S} = \begin{bmatrix} \cos\mu_1 & \sin\mu_1 & 0 & 0 \\ -\sin\mu_1 & \cos\mu_1 & 0 & 0 \\ 0 & 0 & \cos\mu_2 & \sin\mu_2 \\ 0 & 0 & -\sin\mu_2 & \cos\mu_2 \end{bmatrix} \quad .$$

- That yields the expression for the transfer matrix in terms of matrix , $\hat{\mathbf{V}}$

$$\hat{\mathbf{M}} = -\hat{\mathbf{V}}\mathbf{S}\mathbf{U}\hat{\mathbf{V}}^{\mathrm{T}}\mathbf{U} \quad .$$

Transfer Matrix in terms of Twiss Functions

$$\hat{M}_{11} = (1-u)\cos\mu_1 + \alpha_{1x}\sin\mu_1 + u\cos\mu_2 + \alpha_{2x}\sin\mu_2 \quad ,$$

$$\hat{M}_{12} = \beta_{1x}\sin\mu_1 + \beta_{2x}\sin\mu_2 \quad ,$$

$$\hat{M}_{13} = \sqrt{\frac{\beta_{1x}}{\beta_{1y}}}\left[\alpha_{1y}\sin(\mu_1+\nu_1) + u\cos(\mu_1+\nu_1)\right] + \sqrt{\frac{\beta_{2x}}{\beta_{2y}}}\left[\alpha_{2y}\sin(\mu_2-\nu_2) + (1-u)\cos(\mu_2-\nu_2)\right] \quad ,$$

$$\hat{M}_{14} = \sqrt{\beta_{1x}\beta_{1y}}\sin(\mu_1+\nu_1) + \sqrt{\beta_{2x}\beta_{2y}}\sin(\mu_2-\nu_2) \quad ,$$

$$\hat{M}_{21} = -\frac{(1-u)^2 + \alpha_{1x}^2}{\beta_{1x}}\sin\mu_1 - \frac{u^2 + \alpha_{2x}^2}{\beta_{2x}}\sin\mu_2 \quad ,$$

$$\hat{M}_{22} = (1-u)\cos\mu_1 + u\cos\mu_2 - \alpha_{1x}\sin\mu_1 - \alpha_{2x}\sin\mu_2 \quad ,$$

Transfer Matrix in terms of Twiss Functions

$$\hat{M}_{23} = \frac{\left[(1-u)\alpha_{1y} - u\alpha_{1x}\right]\cos(\mu_1+\nu_1) - \left[\alpha_{1x}\alpha_{1y} + u(1-u)\right]\sin(\mu_1+\nu_1)}{\sqrt{\beta_{1x}\beta_{1y}}} + \frac{\left[u\alpha_{2y} - (1-u)\alpha_{2x}\right]\cos(\mu_2-\nu_2) - \left[\alpha_{2x}\alpha_{2y} + u(1-u)\right]\sin(\mu_2-\nu_2)}{\sqrt{\beta_{2x}\beta_{2y}}} \quad ,$$

$$\hat{M}_{24} = \sqrt{\frac{\beta_{1y}}{\beta_{1x}}}\left[(1-u)\cos(\mu_1+\nu_1) - \alpha_{1x}\sin(\mu_1+\nu_1)\right] + \sqrt{\frac{\beta_{2y}}{\beta_{2x}}}\left[u\cos(\mu_2-\nu_2) - \alpha_{2x}\sin(\mu_2-\nu_2)\right] \quad ,$$

$$\hat{M}_{31} = \sqrt{\frac{\beta_{1y}}{\beta_{1x}}}\left[\alpha_{1x}\sin(\mu_1-\nu_1) + (1-u)\cos(\mu_1-\nu_1)\right] + \sqrt{\frac{\beta_{2y}}{\beta_{2x}}}\left[\alpha_{2x}\sin(\mu_2+\nu_2) + u\cos(\mu_2+\nu_2)\right] \quad ,$$

$$\hat{M}_{32} = \sqrt{\beta_{1x}\beta_{1y}}\sin(\mu_1-\nu_1) + \sqrt{\beta_{2x}\beta_{2y}}\sin(\mu_2+\nu_2) \quad ,$$

$$\hat{M}_{33} = u\cos\mu_1 + (1-u)\cos\mu_2 + \alpha_{2y}\sin\mu_2 + \alpha_{1y}\sin\mu_1 \quad ,$$

Transfer Matrix in terms of Twiss Functions

$$\hat{M}_{34} = \beta_{1y} \sin \mu_1 + \beta_{2y} \sin \mu_2 \quad ,$$

$$\hat{M}_{41} = \frac{\left[\alpha_{1x} u - (1-u)\alpha_{1y}\right]\cos(\mu_1 - \nu_1) - \left[\alpha_{1x}\alpha_{1y} + u(1-u)\right]\sin(\mu_1 - \nu_1)}{\sqrt{\beta_{1x}\beta_{1y}}} + \frac{\left[(1-u)\alpha_{2x} - u\alpha_{2y}\right]\cos(\mu_2 + \nu_2) - \left[\alpha_{2x}\alpha_{2y} + u(1-u)\right]\sin(\mu_2 + \nu_2)}{\sqrt{\beta_{2x}\beta_{2y}}} \quad ,$$

$$\hat{M}_{42} = \sqrt{\frac{\beta_{1x}}{\beta_{1y}}}\left[u\cos(\mu_1 - \nu_1) - \alpha_{1y}\sin(\mu_1 - \nu_1)\right] + \sqrt{\frac{\beta_{2x}}{\beta_{2y}}}\left[(1-u)\cos(\mu_2 + \nu_2) - \alpha_{2y}\sin(\mu_2 + \nu_2)\right] ,$$

$$\hat{M}_{43} = -\frac{u^2 + \alpha_{1y}^2}{\beta_{1y}} \sin \mu_1 - \frac{(1-u)^2 + \alpha_{2y}^2}{\beta_{2y}} \sin \mu_2 \quad ,$$

$$\hat{M}_{44} = u \cos \mu_1 + (1-u)\cos \mu_2 - \alpha_{1y} \sin \mu_1 - \alpha_{2y} \sin \mu_2 \quad .$$

Beam ellipsoid in 4D space – bilinear form

$$\hat{\Xi}_{11} = \frac{(1-u)^2 + {\alpha_{1x}}^2}{\varepsilon_1 \beta_{1x}} + \frac{u^2 + {\alpha_{2x}}^2}{\varepsilon_2 \beta_{2x}} \quad ,$$

$$\hat{\Xi}_{22} = \frac{\beta_{1x}}{\varepsilon_1} + \frac{\beta_{2x}}{\varepsilon_2} \quad ,$$

$$\hat{\Xi}_{33} = \frac{u^2 + {\alpha_{1y}}^2}{\varepsilon_1 \beta_{1y}} + \frac{(1-u)^2 + {\alpha_{2y}}^2}{\varepsilon_2 \beta_{2y}} \quad ,$$

$$\hat{\Xi}_{44} = \frac{\beta_{1y}}{\varepsilon_1} + \frac{\beta_{2y}}{\varepsilon_2} \quad ,$$

Beam ellipsoid in 4D space – bilinear form

$$\hat{\Xi}_{12} = \hat{\Xi}_{21} = \frac{\alpha_{1x}}{\varepsilon_1} + \frac{\alpha_{2x}}{\varepsilon_2} \quad ,$$

$$\hat{\Xi}_{34} = \hat{\Xi}_{43} = \frac{\alpha_{1y}}{\varepsilon_1} + \frac{\alpha_{2y}}{\varepsilon_2} \quad ,$$

$$\hat{\Xi}_{13} = \hat{\Xi}_{31} = \frac{\left[\alpha_{1x}\alpha_{1y} + u(1-u)\right]\cos\nu_1 + \left[\alpha_{1y}(1-u) - \alpha_{1x}u\right]\sin\nu_1}{\varepsilon_1\sqrt{\beta_{1x}\beta_{1y}}} + \frac{\left[\alpha_{2x}\alpha_{2y} + u(1-u)\right]\cos\nu_2 + \left[\alpha_{2x}(1-u) - \alpha_{2y}u\right]\sin\nu_2}{\varepsilon_2\sqrt{\beta_{2x}\beta_{2y}}} \quad ,$$

Beam ellipsoid in 4D space – bilinear form

$$\hat{\Xi}_{14} = \hat{\Xi}_{41} = \sqrt{\frac{\beta_{1y}}{\beta_{1x}}}\,\frac{\alpha_{1x}\cos\nu_1 + (1-u)\sin\nu_1}{\varepsilon_1} + \sqrt{\frac{\beta_{2y}}{\beta_{2x}}}\,\frac{\alpha_{2x}\cos\nu_2 - u\sin\nu_2}{\varepsilon_2} \quad ,$$

$$\hat{\Xi}_{23} = \hat{\Xi}_{32} = \sqrt{\frac{\beta_{1x}}{\beta_{1y}}}\,\frac{\alpha_{1y}\cos\nu_1 - u\sin\nu_1}{\varepsilon_1} + \sqrt{\frac{\beta_{2x}}{\beta_{2y}}}\,\frac{\alpha_{2y}\cos\nu_2 + (1-u)\sin\nu_2}{\varepsilon_2} \quad ,$$

$$\hat{\Xi}_{24} = \hat{\Xi}_{42} = \frac{\sqrt{\beta_{1x}\beta_{1y}}\cos\nu_1}{\varepsilon_1} + \frac{\sqrt{\beta_{2x}\beta_{2y}}\cos\nu_2}{\varepsilon_2} \quad .$$

Second order moments in terms of Twiss functions

$$\hat{\mathbf{X}}_{11} \equiv \left\langle x^2 \right\rangle = \varepsilon_1 \beta_{1x} + \varepsilon_2 \beta_{2x} \quad ,$$

$$\hat{\mathbf{X}}_{12} \equiv \left\langle x p_x \right\rangle = \hat{\Sigma}_{21} = -\varepsilon_1 \alpha_{1x} - \varepsilon_2 \alpha_{2x} \; ,$$

$$\hat{\mathbf{X}}_{22} \equiv \left\langle {p_x}^2 \right\rangle = \varepsilon_1 \frac{(1-u)^2 + {\alpha_{1x}}^2}{\beta_{1x}} + \varepsilon_2 \frac{u^2 + {\alpha_{2x}}^2}{\beta_{2x}} \quad ,$$

$$\hat{\mathbf{X}}_{33} \equiv \left\langle y^2 \right\rangle = \varepsilon_1 \beta_{1y} + \varepsilon_2 \beta_{2y} \quad ,$$

$$\hat{\mathbf{X}}_{34} \equiv \left\langle y p_y \right\rangle = \hat{\mathbf{X}}_{43} = -\varepsilon_1 \alpha_{1y} - \varepsilon_2 \alpha_{2y} \; ,$$

$$\hat{\mathbf{X}}_{44} \equiv \left\langle {p_y}^2 \right\rangle = \varepsilon_1 \frac{u^2 + {\alpha_{1y}}^2}{\beta_{1y}} + \varepsilon_2 \frac{(1-u)^2 + {\alpha_{2y}}^2}{\beta_{2y}} \quad ,$$

Second order moments in terms of Twiss functions

$$\hat{\mathbf{X}}_{13} \equiv \langle xy \rangle = \hat{\mathbf{X}}_{31} = \varepsilon_1 \sqrt{\beta_{1x}\beta_{1y}} \cos\nu_1 + \varepsilon_2 \sqrt{\beta_{2x}\beta_{2y}} \cos\nu_2 \quad ,$$

$$\hat{\mathbf{X}}_{14} \equiv \langle x p_y \rangle = \hat{\mathbf{X}}_{41} = \varepsilon_1 \sqrt{\frac{\beta_{1x}}{\beta_{1y}}} \left(u \sin\nu_1 - \alpha_{1y} \cos\nu_1 \right) - \varepsilon_2 \sqrt{\frac{\beta_{2x}}{\beta_{2y}}} \left((1-u)\sin\nu_2 + \alpha_{2y} \cos\nu_2 \right) ,$$

$$\hat{\mathbf{X}}_{23} \equiv \langle y p_x \rangle = \hat{\mathbf{X}}_{32} = -\varepsilon_1 \sqrt{\frac{\beta_{1y}}{\beta_{1x}}} \left((1-u)\sin\nu_1 + \alpha_{1x} \cos\nu_1 \right) + \varepsilon_2 \sqrt{\frac{\beta_{2y}}{\beta_{2x}}} \left(u \sin\nu_2 - \alpha_{2x} \cos\nu_2 \right) \quad ,$$

$$\hat{\mathbf{X}}_{24} \equiv \langle p_x p_y \rangle = \hat{\mathbf{X}}_{42} = \varepsilon_1 \frac{\left(\alpha_{1y}(1-u) - \alpha_{1x} u \right)\sin\nu_1 + \left(u(1-u) + \alpha_{1x}\alpha_{1y} \right)\cos\nu_1}{\sqrt{\beta_{1x}\beta_{1y}}} +$$

$$\varepsilon_2 \frac{\left(\alpha_{2x}(1-u) - \alpha_{2y} u \right)\sin\nu_2 + \left(u(1-u) + \alpha_{2x}\alpha_{2y} \right)\cos\nu_2}{\sqrt{\beta_{2x}\beta_{2y}}} \quad . \quad ,$$

Summary

- Relationships between the eigen-vectors, beam emittances and the beam ellipsoid in 4D phase space
  - From the beam ellipsoid to the eigen-vectors (equivalence of both pictures)
- New parametrization of eigen-vectors in terms of generalized Twiss functions
  - Complete Weyl-like representation
    - 10 independent parameters to fully describe the motion
    - transport line ambiguities resolved
  - Developed software based on this representation allows effective analysis of coupled betatron motion for both circular accelerators and transfer lines (OptiM).