Equations of motion

- Neglecting non-linear effects (such a magnet fringe fields or fields by sextupole magnets), the paraxial Hamiltonian of the transverse motion for a particle with equilibrium energy in an accelerator lattice is given by

$$H[p_x, p_z, x, z; s] = \sum_{q=x,z} \left(\frac{p_q^2}{2} + g_q(s)\frac{q^2}{2} \right).$$

Assuming the coupling to be small, we can consider only one degree of freedom without loss of generality. Hamilton's equations of motion lead to the Hill's equation:

$$\begin{cases} \dot{q} = \partial H/\partial p \\ \dot{p} = -\partial H/\partial q \end{cases} \rightarrow \quad \ddot{q} + g_q(s) q = 0.$$

The general solution for this equation in its standard form is given by

$$q(s) = \sqrt{2 y \beta(s)} \cos(\psi(s) + \delta)$$

$$p(s) = -\alpha(s)\sqrt{\frac{2y}{\beta(s)}} \cos(\psi(s)+\delta) - \sqrt{\frac{2y}{\beta(s)}} \sin(\psi(s)+\delta)$$

where $\beta(s)$ satisfies differential equation $2\beta\ddot{\beta} - \dot{\beta}^2 + 4g(s)\beta^2 = 4$ and

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}, \qquad \alpha(s) = -\dot{\beta}/2.$$

The evolution of the phase-space vector $\zeta(s) = (q(s), p(s))^T$ from azimuthal position s_1 to s_2 is given in terms of transport matrix $\zeta(s_2) = M(s_2|s_1)\zeta(s_1)$, where $M(s_2|s_1)$ is

$$\begin{bmatrix} \sqrt{\frac{\beta(s_2)}{\beta(s_1)}}\left[\cos\mu_{12} + \alpha(s_1)\sin\mu_{12}\right] & \sqrt{\beta(s_1)\beta(s_2)}\,\sin\mu_{12} \\ -\frac{[\alpha(s_2)-\alpha(s_1)]\cos\mu_{12} + [1+\alpha(s_1)\alpha(s_2)]\sin\mu_{12}}{\sqrt{\beta(s_1)\beta(s_2)}} & \sqrt{\frac{\beta(s_1)}{\beta(s_2)}}\left[\cos\mu_{12} - \alpha(s_2)\sin\mu_{12}\right] \end{bmatrix}$$

where

$$\mu_{12} = \int_{s_1}^{s_2} ds/\beta(s).$$

- The use of normalized canonical variables $(p,q) \rightarrow (\mathcal{P},\eta)$

$$\eta = q/\sqrt{\beta} \qquad\qquad \mathcal{P} = p\sqrt{\beta} + \alpha q/\sqrt{\beta}$$

reduces the motion to the simple rotation:

$$\begin{bmatrix} \eta(s_2) \\ \mathcal{P}(s_2) \end{bmatrix} = \begin{bmatrix} \cos\mu_{12} & \sin\mu_{12} \\ -\sin\mu_{12} & \cos\mu_{12} \end{bmatrix} \cdot \begin{bmatrix} \eta(s_1) \\ \eta(s_2) \end{bmatrix}$$

- Another simplification can be performed by introducing normalized complex vector

$$z = \eta - i\mathcal{P} \rightarrow \quad z(s_2) = e^{i\mu_{12}}\, z(s_1)$$

Emittance in different coordinates

- $q_0 = \underbrace{A}_{=\sqrt{2\varepsilon}}\sqrt{\beta}\cos(\underbrace{\psi(0)}_{=0}+\delta) = A\sqrt{\beta}\cos\delta$

$$q_0^2 = A^2\beta\cos^2\delta \quad \rightarrow \quad \langle q_0^2\rangle_\delta = \frac{A^2\beta}{2}$$

- $p_0 = -\alpha\dfrac{A}{\sqrt{\beta}}\cos\delta - \dfrac{A}{\sqrt{\beta}}\sin\delta$

$$p_0^2 = \frac{\alpha^2 A^2}{\beta}\cos^2\delta + \frac{A^2}{\beta}\sin^2\delta + 2\frac{\alpha A^2}{\beta}\sin\delta\cos\delta \rightarrow \langle p_0^2\rangle_\delta = \frac{A^2(1+\alpha^2)}{2\beta}$$

- $q_0 p_0 = -\alpha A^2\cos^2\delta - A^2\sin\delta\cos\delta \quad \rightarrow \quad \langle q_0 p_0\rangle_\delta = -\alpha A^2/2$

$$\varepsilon = \sqrt{\langle q^2\rangle\langle p^2\rangle - \langle qp\rangle^2} = A^2/2 = J$$

- $\eta_0 = A\cos\delta \qquad \eta_0^2 = A^2\cos^2\delta \quad \rightarrow \quad \langle\eta_0^2\rangle = A^2/2$
- $\mathcal{P}_0 = -A\sin\delta \qquad \mathcal{P}_0^2 = A^2\sin^2\delta \quad \rightarrow \quad \langle\mathcal{P}_0^2\rangle = A^2/2$

$$\rightarrow \quad \varepsilon = \sqrt{\langle\eta^2\rangle\langle\mathcal{P}^2\rangle}$$

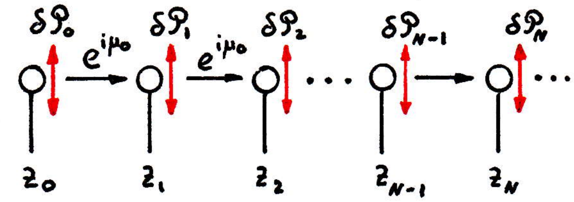
Note that this definition is valid for equilibrium state, otherwise one can use „single particle emittance" and then average over the ensemble

$$\varepsilon = \frac{\langle\eta^2\rangle + \langle\mathcal{P}^2\rangle}{2} \quad \rightarrow \quad \varepsilon = \frac{1}{2}\langle zz^*\rangle$$

Emittance growth due to noise

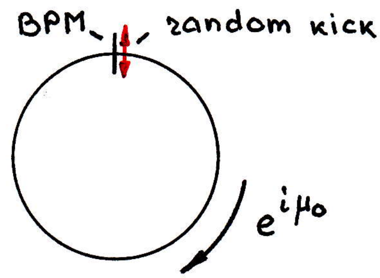
Consider a „turn-by-turn" motion of a particle, when it experiences a thin kick with random nature that changes its momentum as

$$p_n \longrightarrow p_n + \delta p_n$$



Looking for particle behavior right before the location of the random kick source, the result of N successive passes gives

$$z_N = z_0 e^{i\mu_0 N} - i\sum_{n=0}^{N-1} \delta\wp_n e^{i\mu_0(N-n)}$$



where

$$\delta\wp_n = \sqrt{\beta}\,\delta p_n \quad \text{and} \quad \mu_0 = \oint \frac{ds}{\beta(s)}$$

or in terms of normalized coordinates

$$z_N = (\eta_0 - i\wp_0)(\cos\mu_0 N + i\sin\mu_0 N) + \sum_{n=0}^{N-1} \delta\wp_n(\sin\mu_0(N-n) - i\cos\mu_0(N-n))$$

$$\rightarrow \begin{cases} \eta_N = \operatorname{Re} z_N = \eta_0\cos\mu_0 N + \wp_0 \sin\mu_0 N + \sum\limits_{n=0}^{N-1} \delta\wp_n \sin\mu_0(N-n) \\ \wp_N = -\operatorname{Im} z_N = -\eta_0 \sin\mu_0 N + \wp_0\cos\mu_0 N + \sum\limits_{n=0}^{N-1} \delta\wp_n \cos\mu_0(N-n) \end{cases}$$

Or using initial conditions in terms of initial amplitude and phase

$$\eta_0 = \underbrace{\sqrt{2J}}_{=A}\cos(\underbrace{\Psi(0)}_{=0}+\delta) = A\cos\delta \qquad \wp_0 = -\sqrt{2J}\sin(\Psi(0)+\delta) = -A\sin\delta$$

gives

$$\begin{cases} z_N = Ae^{i(\mu_0 N+\delta)} - i\sum\limits_{n=0}^{N-1} \delta\wp_n e^{i\mu_0(N-n)} \\ \eta_N = A\cos(\mu_0 N + \delta) + \sum\limits_{n=0}^{N-1}\delta\wp_n \sin\mu_0(N-n) \\ \wp_N = -A\sin(\mu_0 N+\delta) + \sum\limits_{n=0}^{N-1}\delta\wp_n\cos\mu_0(N-n) \end{cases}$$

Now we can calculate quadratic values of these variables and then average them over the kick value (denoted by $\langle \dots \rangle$). For further calculations we will be assuming that δp_n (and therefore $\delta \wp_n$) is a stationary random process with zero mean value

$\langle \delta p_n \rangle = \langle \delta \wp_n \rangle = 0$

which can be characterized by its auto-correlation function

$\langle \delta\wp_n \delta\wp_m \rangle = \beta \langle \delta p_n \delta p_m \rangle = K_{\delta\wp}(t_1 = Tn,\ t_2 = Tm) = K_{\delta\wp}(\Delta t = t_1 - t_2 = T(n-m))$

where T is the revolution period.

In addition to auto-correlation function one can determine the corresponding spectral density $S(\omega)$:

$$K_{\delta\wp}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{\delta\wp}(\omega) e^{i\omega\tau} d\omega \iff S_{\delta\wp}(\omega) = \int_{-\infty}^{\infty} K_{\delta\wp}(\tau) e^{-i\omega\tau} d\tau$$

such a $S(\omega) \geq 0$ and $S(\omega) = S(-\omega)$.

Therefore

$$\begin{cases} \langle \eta_N^2 \rangle = A^2 \cos^2(\mu_0 N + \delta) + \sum\limits_{n=0}^{N-1}\sum\limits_{m=0}^{N-1} K_{\delta\wp}(T(n-m)) \sin[\mu_0(N-n)] \sin[\mu_0(N-m)] \\ \langle \wp_N^2 \rangle = A^2 \sin^2(\mu_0 N + \delta) + \sum\limits_{n=0}^{N-1}\sum\limits_{m=0}^{N-1} K_{\delta\wp}(T(n-m)) \cos[\mu_0(N-n)] \cos[\mu_0(N-m)] \\ \langle z_N z_N^* \rangle = A^2 + \sum\limits_{n=0}^{N-1}\sum\limits_{m=0}^{N-1} K_{\delta\wp}(T(n-m)) e^{-i\mu_0(n-m)} \end{cases}$$

• $$\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \underbrace{\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega) e^{i\omega T(n-m)} d\omega\right]}_{K_{\delta\wp}(T(n-m))} \sin[\mu_0(N-n)] \sin[\mu_0(N-m)] =$$

$$= \int_{-\infty}^{\infty} d\omega \frac{S(\omega)}{2\pi} \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} e^{i\omega T(n-m)} \frac{e^{i\mu_0(N-n)} - e^{-i\mu_0(N-n)}}{2i} \cdot \frac{e^{i\mu_0(N-m)} - e^{-i\mu_0(N-m)}}{2i} =$$

$$= -\frac{1}{4}\int_{-\infty}^{\infty} d\omega \frac{S(\omega)}{2\pi} \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \Big[e^{i2\mu_0 N} e^{-i(\mu_0 - \omega T)n} e^{-i(\mu_0+\omega T)m} - e^{i(\mu_0+\omega T)n} e^{-i(\mu_0+\omega T)m} + e^{-i2\mu_0 N} e^{i(\mu_0+\omega T)n} e^{i(\mu_0-\omega T)m} - e^{-i(\mu_0-\omega T)n} e^{i(\mu_0-\omega T)m}\Big] =$$

$$= -\frac{1}{4}\int_{-\infty}^{\infty} d\omega \frac{S(\omega)}{2\pi} \Big[\frac{\sin[N(\mu_0-\omega T)/2] \sin[N(\mu_0+\omega T)/2]}{\sin[(\mu_0-\omega T)/2] \sin[(\mu_0+\omega T)/2]} \underbrace{\left(e^{i\mu_0(N+1)} + e^{-i\mu_0(N+1)}\right)}_{2\cos[\mu_0(N+1)]} -$$

$$- \left(\frac{\sin[N(\mu_0-\omega T)/2]}{\sin[(\mu_0-\omega T)/2]}\right)^2 - \left(\frac{\sin[N(\mu_0+\omega T)/2]}{\sin[(\mu_0+\omega T)/2]}\right)^2 \Big] =$$

In the limit $N \to \infty$ the following identities can be used:

$$\begin{cases} \lim\limits_{N\to\infty} \dfrac{\sin N\xi/2}{\sin \xi/2} = 4\pi \sum\limits_{n=-\infty}^{\infty} \delta(\xi - 2\pi n) = 4\pi\, \Delta_{2\pi}(\xi) \\ \lim\limits_{N\to\infty} \dfrac{\sin^2 N\xi/2}{\sin^2 \xi/2} = 2\pi N \sum\limits_{n=-\infty}^{\infty} \delta(\xi - 2\pi n) = 2\pi N\, \Delta_{2\pi}(\xi) \end{cases}$$

$$= -\frac{1}{4}\int_{-\infty}^{\infty} d\omega \frac{S(\omega)}{2\pi}\Big[32\pi^2 \cos[\mu_0(N+1)]\, \underbrace{\Delta_{2\pi}(\mu_0+\omega T)\,\Delta_{2\pi}(\mu_0-\omega T)}_{\equiv 0 \ \forall\, \mu_0 \neq \pi k,\ k\in \mathbb{N}} -$$

$$- 2\pi N\big(\Delta_{2\pi}(\mu_0+\omega T) + \Delta_{2\pi}(\mu_0-\omega T)\big)\Big] =$$

$$= \frac{\pi N}{2}\int_{-\infty}^{\infty} d\omega \frac{S(\omega)}{2\pi}\big(\Delta_{2\pi}(\mu_0+\omega T) + \Delta_{2\pi}(\mu_0-\omega T)\big) =$$ using $\mu_0 = 2\pi\nu$ and revolution frequency $\Omega = 2\pi/T$

$$= \frac{\pi N}{2}\int_{-\infty}^{\infty} d\omega \sum_{n=-\infty}^{\infty}\frac{S(\omega)}{2\pi}\Big(\delta\big[\tfrac{2\pi}{\Omega}(\omega+(\nu-n)\Omega)\big] + \delta\big[\tfrac{2\pi}{\Omega}(\omega-(\nu-n)\Omega)\big]\Big) =$$

$$= \frac{\Omega N}{2}\sum_{n=-\infty}^{\infty} S((\nu-n)\Omega)/2\pi$$

$$\curvearrowright \ \langle \eta_N^2\rangle = A^2\cos^2(\mu_0 N+\delta) + \frac{\Omega N}{2}\sum_{n=-\infty}^{\infty}\frac{S[(\nu-n)\Omega]}{2\pi}$$

(for $\mu_0 = \pi k,\ k\in\mathbb{N}$ $\quad \langle\eta_N^2\rangle = A^2\cos^2(\mu_0 N+\delta)$)

- $$\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\Big[\frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{i\omega T(n-m)}d\omega\Big]\cos[\mu_0(N-n)]\cos[\mu_0(N-m)] =$$

$$= \frac{1}{4}\int_{-\infty}^{\infty} d\omega \frac{S(\omega)}{2\pi}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\Big[e^{i2\mu_0 N}e^{-i(\mu_0-\omega T)n}e^{-i(\mu_0+\omega T)m} + e^{i(\mu_0+\omega T)n}e^{-i(\mu_0+\omega T)m} +$$
$$+ e^{-i2\mu_0 N}e^{i(\mu_0+\omega T)n}e^{i(\mu_0-\omega T)m} + e^{-i(\mu_0-\omega T)n}e^{i(\mu_0-\omega T)m}\Big] =$$

$$= \frac{1}{4}\int_{-\infty}^{\infty} d\omega\frac{S(\omega)}{2\pi}\Big[2\cos[\mu_0(N+1)]\frac{\sin[N(\mu_0-\omega T)/2]\,\sin[N(\mu_0+\omega T)/2]}{\sin[(\mu_0-\omega T)/2]\,\sin[(\mu_0+\omega T)/2]} +$$
$$+ \left(\frac{\sin[N(\mu_0-\omega T)/2]}{\sin[(\mu_0-\omega T)/2]}\right)^2 + \left(\frac{\sin[N(\mu_0+\omega T)/2]}{\sin[(\mu_0+\omega T)/2]}\right)^2\Big] =$$

using the same approximation $N\to\infty$ as for $\langle\eta_N^2\rangle$ gives

$$= \frac{\Omega N}{2}\sum_{n=-\infty}^{\infty} S((\nu-n)\Omega)/2\pi$$

$$\curvearrowright \ \langle \wp_N^2\rangle = A^2\sin^2(\mu_0 N+\delta) + \frac{\Omega N}{2}\sum_{n=-\infty}^{\infty}\frac{S[(\nu-n)\Omega]}{2\pi}$$

$$\bullet \sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, e^{i\omega T(n-m)}\, d\omega\right] e^{-i\mu_0(n-m)} =$$

$$= \int_{-\infty}^{\infty} d\omega \frac{S(\omega)}{2\pi} \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} e^{-i(\mu_0-\omega T)n}\, e^{i(\mu_0-\omega T)m} =$$

$$= \int_{-\infty}^{\infty} d\omega \frac{S(\omega)}{2\pi} \frac{\sin^2[N(\mu_0-\omega T)/2]}{\sin^2[(\mu_0-\omega T)/2]} = \Omega N \sum_{n=-\infty}^{\infty} S((\nu-n)\Omega)/2\pi$$

$$\rightarrow \underline{\lim_{N\to\infty} \langle z_N z_N^* \rangle = A^2 + \Omega N \sum_{n=-\infty}^{\infty} \frac{S((\nu-n)\Omega)}{2\pi}}$$

$$\underline{\underline{\mathcal{E} = \frac{1}{2}\langle z_N z_N^* \rangle = \frac{A^2}{2} + \frac{\Omega N}{2} \sum_{n=-\infty}^{\infty} \frac{S([\nu-n]\Omega)}{2\pi}}}$$

Special case: white noise

The white noise is usually defined as $S(\omega) = \text{const}$, which gives divergency for

$$\sum_{n=-\infty}^{\infty} \frac{S([\nu-n]\Omega)}{2\pi}.$$

Therefore we need to redefine auto-correlation function as

$$K_{\delta\varphi}(T(n-m)) = \langle \delta\varphi^2 \rangle\, \delta_{n,m}$$

where $\delta_{n,m}$ is the Kroneker delta symbol.

Thus (for $\mu \neq \pi k,\ k \in \mathbb{N}$):

$$\bullet \langle \eta_N^2 \rangle = A^2\cos^2(\mu_0 N + \delta) + \langle \delta\varphi^2 \rangle \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \delta_{n,m} \sin[\mu_0(N-n)] \sin[\mu_0(N-m)] =$$

$$= A^2\cos^2(\mu_0 N + \delta) + \langle \delta\varphi^2 \rangle \sum_{n=0}^{N-1} \sin^2[\mu_0(N-n)] =$$

$$= A^2\cos^2(\mu_0 N + \delta) + \frac{\langle \delta\varphi^2 \rangle}{4} \sum_{n=0}^{N-1} \left[2 - e^{i2\mu_0 N} e^{-i2\mu_0 n} - e^{-i2\mu_0 N} e^{i2\mu_0 n}\right] =$$

$$= \underline{A^2\cos^2(\mu_0 N + \delta) + \frac{\langle \delta\varphi^2 \rangle N}{2} - \cos[\mu_0(N+1)] \frac{\sin \mu_0 N}{\sin \mu_0} \frac{\langle \delta\varphi^2 \rangle}{2}}$$

$$\bullet \langle \varphi_N^2 \rangle = A^2\sin^2(\mu_0 N + \delta) + \langle \delta\varphi^2 \rangle \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \delta_{n,m} \cos[\mu_0(N-n)] \cos[\mu_0(N-m)] =$$

$$= \underline{A^2\sin^2(\mu_0 N + \delta) + \frac{\langle \delta\varphi^2 \rangle N}{2} + \cos[\mu_0(N+1)] \frac{\sin \mu_0 N}{\sin \mu_0} \frac{\langle \delta\varphi^2 \rangle}{2}}$$

- $\langle z_N z_N^* \rangle = A^2 + \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \langle \delta \wp^2 \rangle e^{-i\mu_0 (n-m)} \delta_{n,m} = A^2 + \langle \delta\wp^2 \rangle \sum_{n=0}^{N-1} 1 = \underline{A^2 + \langle \delta \wp^2 \rangle N}$

$$\rightarrow \underline{\underline{\mathcal{E} = \frac{1}{2} \langle z_N z_N^* \rangle = \frac{A^2}{2} + \frac{\langle \delta\wp^2\rangle N}{2}}}$$

For the resonant case:

$$\begin{cases} \langle \eta_N^2 \rangle = A^2 \cos^2(\mu_0 N + \delta) & \rightarrow \langle \overline{\eta_N^2} \rangle = A^2/2 \\ \langle \wp_N^2 \rangle = A^2 \sin^2(\mu_0 N + \delta) + \langle \delta\wp^2 \rangle N & \rightarrow \langle \overline{\wp_N^2} \rangle = A^2/2 + \langle \delta\wp^2\rangle N \\ \mathcal{E} = \dfrac{\langle \overline{\eta}_N^2 \rangle + \langle \overline{\wp}_N^2 \rangle}{2} = \dfrac{A^2}{2} + \dfrac{\langle \delta\wp^2 \rangle N}{2} \end{cases}$$

Cross-correlation

$$\langle \eta_N \wp_N \rangle = -A^2 \sin(\mu_0 N + \delta)\cos(\mu_0 N + \delta) + \langle \delta\wp^2 \rangle \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \delta_{n,m} \sin[\mu_0 (N-n)] \times$$

$$\times \cos[\mu_0 (N-m)] =$$

$$= -A^2 \sin(\mu_0 N + \delta)\cos(\mu_0 N+\delta) + \langle \delta\wp^2 \rangle \sum_{n=0}^{N-1} \frac{1}{2} \sin[2\mu_0 (N-n)] =$$

$$= -\frac{A^2}{2}\sin[2(\mu_0 N + \delta)] + \frac{1}{4i}\langle \delta\wp^2\rangle \sum_{n=0}^{N-1} \left(e^{i2\mu_0 N} e^{-i2\mu_0 n} - \text{c.c.}\right) =$$

$$= -\frac{A^2}{2}\sin[2(\mu_0 N + \delta)] + \underbrace{\frac{\langle \delta\wp^2 \rangle}{2} \frac{\sin\mu_0 N}{\sin\mu_0} \sin[\mu_0 (N+1)]}_{\text{this term }=0\text{ for the resonant case}}$$

$\rightarrow$

$$\langle \overline{\eta_N \wp_N} \rangle = \frac{\langle \delta\wp^2 \rangle}{2} \frac{\sin\mu_0 N}{\sin\mu_0} \sin[\mu_0 (N+1)]$$

$$\rightarrow \mathcal{E} = \sqrt{\langle \overline{\eta}_N^2 \rangle \langle \overline{\wp}_N^2 \rangle - \langle \overline{\eta_N \wp_N} \rangle^2} = \left[\left(\frac{A^2}{2} + \frac{\langle \delta\wp^2 \rangle N}{2}\right)^2 - \underbrace{\frac{\langle \delta\wp^2\rangle^2}{4} \frac{\sin^2\mu_0 N}{\sin^2 \mu_0}}_{\xrightarrow[N\to\infty,\ \mu_0 \neq \pi k]{} 0}\right]^{1/2} =$$

$$= \frac{A^2}{2} + \frac{\langle \delta\wp^2 \rangle N}{2}$$

in resonant case

$$\mathcal{E} = \sqrt{\frac{A^2}{2}\left(\frac{A^2}{2} + \langle \delta\wp^2 \rangle N\right)}$$

Preliminary consideration of the choice of the coefficients

In most general form the filter which is based on a prehistory of N turns and applied d turns later can be represented as:

$$\delta p_n = \frac{g}{\sqrt{\beta_p \beta_k}} \sum_{k=0}^{N-1} A_k x_{n-d-k}$$

where n is the turn # where the filter output should be used.

- The filter shouldn't be sensitive to the orbit offset with respect to the geometrical center of BPM:

$$\text{for } x_n = \text{const} = \alpha \qquad \sum_{k=0}^{N-1} A_k x_{n-d-k} = \alpha \sum_{k=0}^{N-1} A_k = 0$$

$$\rightarrow \sum_{k=0}^{N-1} A_k = 0 \quad \text{- notch filter}$$

- In addition to notch filtering one can make the filter to be non-sensitive to the linear change of the orbit in time:

$$\text{for } x_n = \beta n \qquad \sum_{k=0}^{N-1} A_k x_{n-d-k} = \beta \sum_{k=0}^{N-1} A_k (n-d-k) = \beta(n-d) \sum_{k=0}^{N-1} A_k - \beta \sum_{k=0}^{N-1} k A_k = 0$$

$$\rightarrow \begin{cases} \sum_{k=0}^{N-1} A_k = 0 \\ \sum_{k=0}^{N-1} k A_k = 0 \end{cases}$$

- These considerations can be generalized in order to make the filter to be non-sensitive to the polynomial orbit changing in time:

$$x_n = \gamma n^m \quad \rightarrow \quad m+1 \left\{ \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & (m+1)^0 \\ 0 & 1 & 2 & 3 & \cdots & (m+1)^1 \\ 0 & 1 & 4 & 9 & \cdots & (m+1)^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2^m & 3^m & \cdots & (m+1)^m \end{bmatrix}}_{m+2} \cdot \begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_{m+1} \end{bmatrix} = 0 \right.$$

$$\rightarrow \quad A_k = (-1)^k C_k^{m+1}, \quad k = 0, 1, \ldots, m+1$$

m	N	A_0	A_1	A_2	A_3	A_4	A_5
0	2	1	-1	–	–	–	–
1	3	1	-2	1	–	–	–
2	4	1	-3	3	-1	–	–
3	5	1	-4	6	-4	1	–
4	6	1	-5	10	-10	5	-1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Modeling of the transverse damper

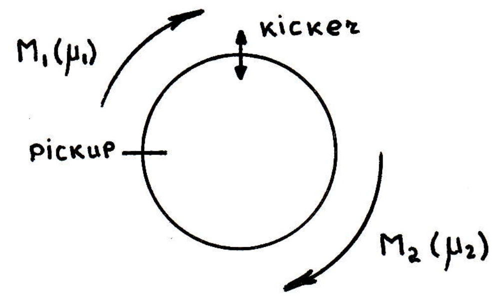
Equations of motion

One turn map is given as follows:

$$\begin{bmatrix} x_n \\ p_n \end{bmatrix}_P \to \begin{bmatrix} x_n \\ p_n \end{bmatrix}_{K-0} = M_1 \begin{bmatrix} x_n \\ p_n \end{bmatrix}_P \to \begin{bmatrix} x_n \\ p_n \end{bmatrix}_{K+0} = \begin{bmatrix} x_n \\ p_n \end{bmatrix}_{K-0} + \begin{bmatrix} 0 \\ \delta p_n \end{bmatrix} \to \begin{bmatrix} x_{n+1} \\ p_{n+1} \end{bmatrix}_P = M_2 \begin{bmatrix} x_n \\ p_n \end{bmatrix}_{K+0}$$

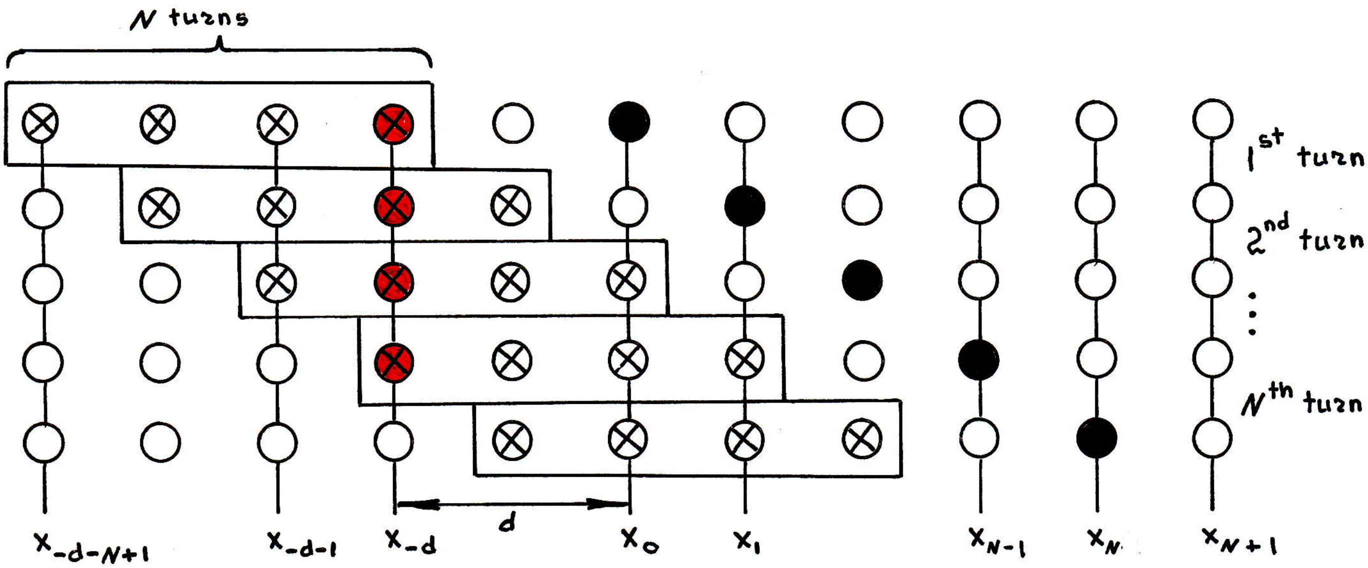
or ommiting subscript „P"

$$\begin{bmatrix} x_{n+1} \\ p_{n+1} \end{bmatrix} = M_2 \left(M_1 \begin{bmatrix} x_n \\ p_n \end{bmatrix} + \begin{bmatrix} 0 \\ \delta p_n \end{bmatrix} \right)$$



where

$$\delta p_n = \frac{g}{\sqrt{\beta_P \beta_K}} \sum_{k=0}^{N-1} A_k \left(x_{n-d-k} + \delta x_{n-d-k} \right)$$



Using normalized complex variable $z = \eta - i\mathcal{P}$:

$$z_{n+1} = e^{i\mu_2} \left(e^{i\mu_1} z_n - i\,\delta\mathcal{P}_n \right)$$

where

$$\delta\mathcal{P}_n = \sqrt{\beta_K}\,\delta p_n = g \sum_{k=0}^{N-1} A_k \left(\operatorname{Re} z_{n-d-k} + \delta\eta_{n-d-k} \right)$$

$$\delta\eta_i = \delta x_i / \sqrt{\beta_P}$$

So finally

$$z_{n+1} = e^{i\mu_2} \left(e^{i\mu_1} z_n - ig \sum_{k=0}^{N-1} A_k \left[\frac{z_{n-d-k} + z^*_{n-d-k}}{2} + \delta\eta_{n-d-k} \right] \right)$$

Damping rate for small gain

Neglecting heating term $\delta\eta_i$, and taking into account that addends contribution of z^* term is averaged out, gives:

$$z_{n+1} = e^{i\mu_0}\left[z_n - i\frac{g}{2} e^{-i\mu_1} \sum_{k=0}^{\mathcal{K}-1} A_k z_{n-d-k} \right].$$

Looking for the solution in the form $z_n = z_0 e^{i\mu n}$ $\rightarrow$

$$e^{i(\mu-\mu_0)} = 1 - i\frac{g}{2} e^{-i(\mu_1+\mu d)} \sum_{k=0}^{\mathcal{K}-1} A_k e^{-i\mu k}$$

Then, using the first-order perturbation theory $\mu = \mu_0 + i g_d$

$$g_d = i\frac{g}{2} e^{-i(\mu_1+\mu_0 d)} \sum_{k=0}^{\mathcal{K}-1} A_k e^{-i\mu_0 k}$$

$\rightarrow$ Emittance damping can be calculated as

$$\varepsilon = \langle z z^* \rangle / 2 = \frac{\langle z_0^2 \rangle}{2} e^{i(\mu_0 - \mathrm{Im}\, g_d)n} e^{-i(\mu_0 - \mathrm{Im}\, g_d)n} e^{-2\mathrm{Re}\, g_d} = \varepsilon_0 e^{-2\mathrm{Re}\, g_d}$$

In order to have critically damped system $\mathrm{Im}\, g_d$ should be equal to zero, which gives

$$\sum_{k=0}^{\mathcal{K}-1} A_k \cos[\mu_0 k + \delta\mu] = 0 \quad , \text{where } \delta\mu = \mu_1 + \mu_0 d$$

Emittance growth excited by noise of BPM

Now, keeping only the heating term, while damping term will be omitted, a one turn map is given by

$$z_{n+1} = e^{i\mu_0}\left[z_n - i g e^{-i\mu_1} \sum_{k=0}^{\mathcal{K}-1} A_k \delta\eta_{n-d-k} \right]$$

If only one measurement is erroneous, let say $\delta\eta_{-d}$, then after $\mathcal{K}$ turns:

$$z_{\mathcal{K}} = z_0 e^{i\mu_0\mathcal{K}} - i g\, \delta\eta_{-d}\, e^{-i\mu_1} \sum_{k=0}^{\mathcal{K}-1} A_k e^{i\mu_0(\mathcal{K}-k)} = (z_0 - 2 g_d\, \delta\eta_{-d}\, e^{i\mu_0 d}) e^{i\mu_0\mathcal{K}}$$

$$\rightarrow \quad \delta\varepsilon = \langle \overline{\delta z \delta z^*} \rangle / 2 = 2\overline{\delta\eta^2}\left(\mathrm{Re}^2 g_d + \mathrm{Im}^2 g_d \right) = 2|g_d|^2\, \overline{\delta q^2} / \beta_p$$

where we used additional averaging over kick amplitudes $\overline{(\ldots)}$, and

$(\overline{\delta q^2})^{1/2} = \sqrt{\beta_p}\,(\overline{\delta\eta^2})^{1/2}$ is the RMS error of a single measurement

Feedback System

Consider the simplest case when $\mathcal{K}=0$ and $d=0$:

$$\begin{bmatrix}\eta_{n+1}\\ \wp_{n+1}\end{bmatrix}=\widetilde{M}_2\left(\widetilde{M}_1\begin{bmatrix}\eta_n\\ \wp_n\end{bmatrix}+\begin{bmatrix}0\\ \delta\wp_n\end{bmatrix}\right), \qquad \text{where } \delta\wp_n=g\eta_n$$

This equation can be rewritten as

$$\begin{bmatrix}\eta_{n+1}\\ \wp_{n+1}\end{bmatrix}=\begin{bmatrix}\cos\mu_2 & \sin\mu_2\\ -\sin\mu_2 & \cos\mu_2\end{bmatrix}\cdot\left(\begin{bmatrix}\cos\mu_1 & \sin\mu_1\\ -\sin\mu_1 & \cos\mu_1\end{bmatrix}+\begin{bmatrix}0 & 0\\ g & 0\end{bmatrix}\right)\cdot\begin{bmatrix}\eta_n\\ \wp_n\end{bmatrix}=$$

$$=\begin{bmatrix}g\sin\mu_2+\cos(\mu_1+\mu_2) & \sin(\mu_1+\mu_2)\\ g\cos\mu_2-\sin(\mu_1+\mu_2) & \cos(\mu_1+\mu_2)\end{bmatrix}\cdot\begin{bmatrix}\eta_n\\ \wp_n\end{bmatrix}=\widetilde{M}\begin{bmatrix}\eta_n\\ \wp_n\end{bmatrix}$$

Characteristic polynomial for $\widetilde{M}$ is

$$\lambda^2-(2\cos(\mu_1+\mu_2)+g\sin\mu_2)\lambda+1-g\sin\mu_1=0$$

The discriminant of this polynomial is

$$\Delta_\lambda=g^2\sin^2\mu_2+4(\sin\mu_1+\sin\mu_2\cos(\mu_1+\mu_2))\,g-4$$

If $\mu_2\neq\pi m$, $m=0,1,2,\dots$, then

for $\sin(\mu_1+\mu_2)>0$ $\begin{cases}\Delta_\lambda>0 & \text{if } g\in\left(-\infty;-\frac{\sin(\mu_1+\mu_2)}{\sin^2(\mu_2/2)}\right)\cup\left(\frac{\sin(\mu_1+\mu_2)}{\cos^2(\mu_2/2)};\infty\right)\\ \Delta_\lambda\le 0 & \text{if } g\in\left[-\frac{\sin(\mu_1+\mu_2)}{\sin^2(\mu_2/2)};\frac{\sin(\mu_1+\mu_2)}{\cos^2(\mu_2/2)}\right]\end{cases}$

If $\Delta_\lambda\le 0$ then $\lambda_{1,2}$ have equal modulus

$$\lambda_{1,2}=\frac{g}{2}\sin\mu_2+\cos(\mu_1+\mu_2)\pm i\sqrt{1-g\sin\mu_1-\left[\frac{g}{2}\sin\mu_2+\cos(\mu_1+\mu_2)\right]^2}$$

$|\lambda_{1,2}|=\sqrt{1-g\sin\mu_1}$ which is <1 for $\begin{cases}g\in(0;\sin^{-1}\mu_1), & \sin\mu_1>0\\ g\in(\sin^{-1}\mu_1;0), & \sin\mu_1<0\end{cases}$

The damping decrement is given as

$$f_0\min\{(1-|\lambda_1|),(1-|\lambda_2|)\}$$

Calculation of coefficients

$$A_k = (-1)^k \left[C_k^{N+1} \sin\left(\frac{N+3}{2}\mu_0 + \delta\mu + N\frac{\pi}{2}\right) + C_{k-1}^{N+1} \sin\left(\frac{N+1}{2}\mu_0 + \delta\mu + N\frac{\pi}{2}\right)\right]$$

Notation :

$\boxed{N}$	filter type	{0 - notch, 1 - linear change, 2 - quad. change}
$\mathcal{K}$	# of turns	$\mathcal{K} = N+3$
k	index	$k \in [0, N+2]$
$\boxed{\mu_1}$ exp.	pickup - kicker ph. adv.	
μ_2	kicker - pickup ph. adv.	
μ_0	Betatron tune	$\mu_0 = \mu_1 + \mu_2 = 2\pi \boxed{\nu_0}$
$\boxed{d}$ exp.	# of delayed turns	in our case $d = 0$
$\delta\mu$		$\delta\mu = \mu_1 + \mu_0 d$

„d" should be protected, $d = 0$

„μ_1" should be protected and placed into „expert mode page".

$$\mu_{1x} = \frac{11}{12}\, 2\pi \nu_{0x} \quad \text{- horizontal}$$

$$\mu_{1y} \approx \frac{239}{240}\, 2\pi \nu_{0y} \quad \text{- vertical}$$

Check of decrement sign

$$g_d = i\frac{g}{2} e^{-i(\mu_1 + \mu_0 d)} \sum_{k=0}^{\mathcal{K}-1} A_k e^{-i\mu_0 k} = \underbrace{\frac{g}{2}\sum_{k=0}^{\mathcal{K}-1} A_k \sin(\mu_0 k + \delta\mu)}_{\text{Re } g_d \text{ should be } >0\text{, otherwise } A_k \to -A_k} + \underbrace{i\frac{g}{2}\sum_{k=0}^{\mathcal{K}-1} A_k \cos(\mu_0 k + \delta\mu)}_{\text{Im } g_d \text{ should be equal to } 0.}$$

Possible Normalization

$$\sum_{k=0}^{\mathcal{K}-1} A_k^2 = 1 \;\curvearrowright\; A_k \to A_k \Big/ \sqrt{\sum_{k=0}^{\mathcal{K}-1} A_k^2}$$

Sum rearrangment

In order to reduce the number of coefficients, one can rearrange the sum as follows:

- Notch filter $(m=0)$

$$\sum_{k=0}^{N-1} A_k x_{(n-d)-k} = A_0 x_{(n-d)} + \underbrace{A_0\left[x_{(n-d)-1} - x_{(n-d)-1}\right]}_{\equiv 0} + A_1 x_{(n-d)-1} + \underbrace{(A_0+A_1)\left[x_{(n-d)-2} - x_{(n-d)-2}\right]}_{\equiv 0} +$$

$$+ A_2 x_{(n-d)-2} + \underbrace{(A_0+A_1+A_2)\left[x_{(n-d)-3} - x_{(n-d)-3}\right]}_{\equiv 0} + \dots + A_{N-1} x_{(n-d)-(N-1)} =$$

$$= (A_0)\left[x_{(n-d)} - x_{(n-d)-1}\right] + (A_0+A_1)\left[x_{(n-d)-1} - x_{(n-d)-2}\right] + (A_0+A_1+A_2)\left[x_{(n-d)-2} - x_{(n-d)-3}\right] +$$

$$+ \dots + \left(\sum_{i=0}^{N-2} A_i\right)\left[x_{(n-d)-(N-2)} - x_{(n-d)-(N-1)}\right] + \left(\sum_{i=0}^{N-1} A_i\right) x_{(n-d)-(N-1)} =$$

$$= \sum_{k=0}^{N-2} B_k \left(x_{(n-d)-k} - x_{(n-d)-(k+1)}\right) \quad \text{where} \quad B_k = \sum_{i=0}^{k} A_i$$

- Correction of linear change of orbit in time $(m=1)$

$$\sum_{k=0}^{N-2} B_k \left(x_{(n-d)-k} - x_{(n-d)-(k+1)}\right) = \sum_{k=0}^{N-3} C_k \left(x_{(n-d)-k} - 2x_{(n-d)-(k+1)} + x_{(n-d)-(k+2)}\right)$$

where $C_k = \sum_{i=0}^{k} B_i$

- This procedure can be generalized for arbitrary m:

$$\sum_{k=0}^{N-1} A_k x_{(n-d)-k} = \sum_{k=0}^{N-m-2} \left[D_k \sum_{l=0}^{m+1} \left((-1)^l C_l^{m+1} x_{(n-d)-(k+l)} \right) \right]$$

where $D_k = \underbrace{\sum_{l=0}^{k} \sum_{i=0}^{l} \dots \sum_{z=0}^{q}}_{(m+1)\ \text{sums}} A_z$