Accelerator Physics

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Lecture 9

Synchrotron Radiation

Accelerated particles emit electromagnetic radiation. Emission from very high energy particles has unique properties for a radiation source. As such radiation was first observed at one of the earliest electron synchrotrons, radiation from high energy particles (mainly electrons) is known generically as synchrotron radiation by the accelerator and HENP communities.

The radiation is highly collimated in the beam direction

From relativity

$$ct' = \gamma ct - \gamma\beta z$$

$$x' = x$$

$$y' = y$$

$$z' = -\gamma\beta ct + \gamma z$$

Lorentz invariance of wave phase implies $k^\mu = (\omega/c, k_x, k_y, k_z)$ is a Lorentz 4-vector

$$\omega' = \gamma\omega - \gamma\beta k_z c$$
$$k_x' = k_x$$
$$k_y' = k_y$$
$$k_z' = -\gamma\beta c\omega + \gamma k_z$$

$$\sin\theta = \frac{\sqrt{k_x^2 + k_y^2}}{\omega / c} \qquad \sin\theta' = \frac{\sqrt{k_x'^2 + k_y'^2}}{\omega' / c} \qquad \cos\theta' = \frac{k_z'}{\omega' / c}$$

$$\omega / c = \gamma\omega' / c + \gamma\beta k_z' = \gamma\left(1 + \beta\cos\theta'\right)\left(\omega' / c\right)$$

$$\theta \approx \sin\theta = \frac{\sin\theta'}{\gamma\left(1+\beta\cos\theta'\right)}$$

Therefore all radiation with $\theta' < \pi / 2$, which is roughly ½ of the emission for dipole emission from a transverse acceleration in the beam frame, is Lorentz transformed into an angle less than $1/\gamma$. Because of the strong Doppler shift of the photon energy, higher for $\theta \to 0$, most of the energy in the photons is within a cone of angular extent $1/\gamma$ around the beam direction.

Larmor's Formula

For a particle executing non-relativistic motion, the total power emitted in electromagnetic radiation is (Larmor)

$$P(t) = \frac{1}{6\pi\varepsilon_0}\frac{q^2}{c^3}\left|\vec{a}\right|^2 = \frac{1}{6\pi\varepsilon_0}\frac{e^2}{m^2c^3}\left|\dot{\vec{p}}\right|^2$$

Lienard's relativistic generalization: Note both *dE* and *dt* are the fourth component of relativistic 4-vectors when one is dealing with photon emission. Therefore, their ratio must be an Lorentz invariant. The invariant that reduces to Larmor's formula in the non-relativistic limit is

$$P = -\frac{e^2}{6\pi\varepsilon_0 c}\frac{du^\mu}{d\tau}\frac{du_\mu}{d\tau}$$

$$P(t) = \frac{e^2}{6\pi\varepsilon_0 c}\gamma^6\left(\dot{\vec{\beta}}^2 - \left[\vec{\beta}\times\dot{\vec{\beta}}\right]^2\right)$$

For acceleration along a line, second term is zero and first term for the radiation reaction is small compared to the acceleration as long as gradient less than 10^{14} MV/m. Technically impossible.

For transverse bend acceleration $\dot{\vec{\beta}} = -\frac{\beta^2 c}{\rho}\hat{r}$

$$P(t) = \frac{e^2 c}{6\pi\varepsilon_0\rho^2}\beta^4\gamma^4$$

Fractional Energy Loss

$$\delta E = \frac{e^2}{6\pi\varepsilon_0\rho}\Theta\beta^3\gamma^4$$

For one turn with isomagnetic bending fields

$$\frac{\delta E}{E_{beam}} = \frac{4\pi r_e}{3\rho}\beta^3\gamma^3$$

r_e is the classical electron radius: 2.82×10^{-13} cm

Radiation Power Distribution

Consulting your favorite Classical E&M text (Jackson, Schwinger, Landau and Lifshitz Classical Theory of Fields)

$$\frac{dP}{d\omega} = \frac{\sqrt{3}}{8\pi^2 \varepsilon_0} \frac{e^2}{\rho} \gamma \frac{\omega}{\omega_c} \int_{\omega/\omega_c}^{\infty} K_{5/3}(x) dx$$

Critical Frequency

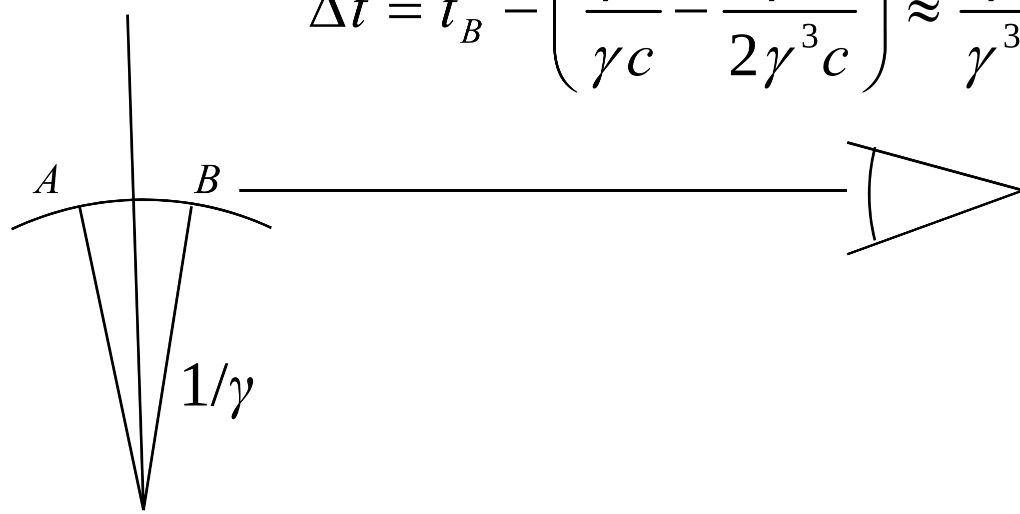
Critical (angular) frequency is

$$\omega_c = \frac{3}{2}\gamma^3 \frac{c}{\rho}$$

Energy scaling of critical frequency is understood from $1/\gamma$ emission cone and fact that $1-\beta \sim 1/(2\gamma^2)$

$$\Delta t = t_B - \left(\frac{\rho}{\gamma c} - \frac{\rho}{2\gamma^3 c} \right) \approx \frac{\rho}{\gamma^3 c}$$

$$t_A = -\frac{\rho}{\gamma\beta c}$$



$$t_B = \frac{\rho}{\gamma\beta c} \approx \frac{\rho}{\gamma c} + \frac{\rho}{2\gamma^3 c}$$

Photon Number

$$P=\int_0^\infty \frac{dP}{d\omega}d\omega=\frac{\sqrt{3}}{8\pi^2\varepsilon_0}\frac{e^2}{\rho}\omega_c\gamma\int_0^\infty \xi\int_\xi^\infty K_{5/3}\left(x\right)dx\,d\xi=\frac{e^2c}{6\pi\varepsilon_0\rho^2}\gamma^4$$

$$\frac{d\dot{n}}{d\omega}=\frac{1}{\hbar\omega}\frac{dP}{d\omega}$$

$$\left\langle\hbar\omega\right\rangle=\frac{\int_0^\infty \hbar\omega\frac{d\dot{n}}{d\omega}d\omega}{\int_0^\infty \frac{d\dot{n}}{d\omega}d\omega}=\frac{8}{15\sqrt{3}}\hbar\omega_c$$

$$\dot{n}=\frac{5\alpha}{2\sqrt{3}}\frac{c}{\rho}\gamma \qquad \delta n=\frac{5\alpha}{2\sqrt{3}}\Theta\gamma \qquad \alpha=\frac{e^2}{4\pi\varepsilon_0\hbar c}\approx\frac{1}{137}$$

Insertion Devices

Often periodic magnetic field magnets are placed in beam path of high energy storage rings. The radiation generated by electrons passing through such insertion devices has unique properties.

Field of the insertion device magnet

$$\vec{B}(x,y,z) = B(z)\,\hat{y} \qquad B(z) \approx B_0 \cos\left(2\pi z/\lambda_{ID}\right)$$

Vector potential for magnet (1 dimensional approximation)

$$\vec{A}(x,y,z) = A(z)\,\hat{x} \qquad A(z) \approx \frac{B_0 \lambda_{ID}}{2\pi} \sin\left(2\pi z/\lambda_{ID}\right)$$

Electron Orbit

Uniformity in x-direction means that canonical momentum in the x-direction is conserved

$$\mathrm{v}_x(z) = \frac{eA(z)}{\gamma m} = \frac{K}{\gamma} c \sin\left(2\pi z / \lambda_{ID}\right)$$

$$x(z) = \int \frac{\mathrm{v}_x}{\mathrm{v}_z} dz \approx -\frac{1}{\langle \beta_z \rangle} \frac{K}{\gamma} \frac{\lambda_{ID}}{2\pi} \cos\left(2\pi z / \lambda_{ID}\right)$$

Field Strength Parameter

$$K \equiv \frac{eB_0 \lambda_{ID}}{2\pi mc}$$

Average Velocity

Energy conservation gives that γ is a constant of the motion

$$\beta_z(z) = \sqrt{1 - \frac{1}{\gamma^2} - \beta_x^2(z)}$$

Average longitudinal velocity in the insertion device is

$$\beta^{*2} = \langle \beta_z \rangle^2 = 1 - \frac{1}{\gamma^2} - \frac{K^2}{2\gamma^2}$$

Average rest frame has

$$\gamma^{*2} = \frac{1}{1 - \beta^{*2}} = \frac{\gamma^2}{1 + K^2/2}$$

Relativistic Kinematics

In average rest frame the insertion device is Lorentz contracted, and so its wavelength is

$$\lambda^* = \lambda_{ID} / \beta^* \gamma^*$$

The sinusoidal wiggling motion emits with angular frequency

$$\omega^* = 2\pi c / \lambda^*$$

Lorentz transformation formulas for the wave vector

$$k^* = \gamma^* k\left(1 - \beta^* \cos\theta\right)$$
$$k_x^* = k_x = k \sin\theta \cos\varphi$$
$$k_y^* = k_y = k \sin\theta \sin\varphi$$
$$k_z^* = \gamma^* k\left(\cos\theta - \beta^*\right)$$

Insertion Device (FEL) Resonance Condition

Angle transforms as

$$\cos\theta^* = \frac{k_z^*}{k^*} = \frac{\left(\cos\theta - \beta^*\right)}{\left(1-\beta^*\cos\theta\right)}$$

Wave vector in lab frame has

$$k = \frac{k^*}{\gamma^*\left(1-\beta^*\cos\theta\right)} = \frac{2\pi\beta^* c}{\lambda_{ID}\left(1-\beta^*\cos\theta\right)}$$

In the forward direction $\cos\theta = 1$

$$\lambda_e \approx \frac{\lambda_{ID}}{2\gamma^{*2}} = \frac{\lambda_{ID}}{2\gamma^2}\left(1+K^2/2\right)$$

Power Emitted Lab Frame

Larmor/Lienard calculation in the lab frame yields

$$\langle P \rangle = \frac{e^2}{6\pi\varepsilon_0}\gamma^4\beta^{*2}c\left(\frac{K}{\gamma}\right)^2\left(\frac{2\pi}{\lambda_{ID}}\right)^2\frac{1}{2}$$

Total energy radiated after one passage of the insertion device

$$\delta E = 2\pi^2\frac{e^2}{6\pi\varepsilon_0\lambda_{ID}}\gamma^2\beta^*NK^2$$

Power Emitted Beam Frame

Larmor/Lienard calculation in the beam frame yields

$$\left\langle P^* \right\rangle = \frac{2e^2}{6\pi\varepsilon_0} cK^2 \left(\frac{2\pi}{\lambda^*}\right)^2 \frac{1}{2}$$

Total energy of each photon is $\hbar 2\pi c/\lambda^*$, therefore the total number of photons radiated after one passage of the insertion device

$$N_\gamma = \frac{2\pi}{3}\alpha N K^2$$

Spectral Distribution in Beam Frame

Begin with power distribution in beam frame: dipole radiation pattern (single harmonic only when K<<1; replace γ* by γ, β* by β)

$$\frac{dP^*}{d\Omega^*} = \frac{e^2 c}{32\pi^2 \varepsilon_0} k^{*4} a^2 \sin^2 \Theta^*$$

Number distribution in terms of wave number

$$\frac{dN_\gamma}{d\Omega^*} = \frac{\alpha}{4} N K^2 \frac{k_y^{*2} + k_z^{*2}}{k^{*2}}$$

Solid angle transformation

$$d\Omega^* = \frac{d\Omega}{\gamma^2 \left(1 - \beta \cos\theta\right)^2}$$

Number distribution in beam frame

$$\frac{dN_\gamma}{d\Omega} = \frac{\alpha}{4} NK^2 \frac{\sin^2\theta \sin^2\varphi + \gamma^2 \left(\cos\theta - \beta\right)^2}{\gamma^4 \left(1-\beta\cos\theta\right)^4}$$

Energy is simply

$$E\left(\theta\right) = \hbar \frac{2\pi\beta c}{\lambda_{ID}\left(1-\beta\cos\theta\right)} \qquad \hat{E}\left(\theta\right) = \frac{1}{\left(1-\beta\cos\theta\right)}$$

Number distribution as a function of normalized lab-frame energy

$$\frac{dN_\gamma}{d\hat{E}} = \frac{\alpha\pi}{4\gamma^2\beta^3} NK^2 \left[\left(\frac{\hat{E}}{\gamma^2} - 1\right)^2 + \beta^2\right]$$

Average Energy

Limits of integration

$$\cos\theta = 1 \qquad \hat{E} = \frac{1}{1-\beta} \qquad \cos\theta = -1 \qquad \hat{E} = \frac{1}{1+\beta}$$

Average energy is also analytically calculable

$$\langle E \rangle = \frac{\int_0^\infty E \frac{dN_\gamma}{d\hat{E}} d\hat{E}}{\int_0^\infty \frac{dN_\gamma}{d\hat{E}} d\hat{E}} = \gamma^2 \hbar 2\pi\beta c / \lambda_{ID} \approx \frac{E_{\max}}{2}$$

Conventions on Fourier Transforms

For the time dimensions

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-i\omega t} d\omega$$

Results on Dirac delta functions

$$\tilde{\delta}(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{i\omega t} dt = 1$$

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} d\omega$$

For the three spatial dimensions

$$\tilde{f}\left(\vec{k}\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(\vec{x}\right)e^{-i\vec{k}\cdot\vec{x}}d^{3}\vec{x}$$

$$f\left(\vec{x}\right)=\frac{1}{\left(2\pi\right)^{3}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}\left(\vec{k}\right)e^{+i\vec{k}\cdot\vec{x}}d^{3}\vec{k}$$

$$\delta^{3}\left(\vec{x}\right)=\delta\left(\vec{x}\right)=\frac{1}{\left(2\pi\right)^{3}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{+i\vec{k}\cdot\vec{x}}d^{3}\vec{k}$$

Green Function for Wave Equation

Solution to inhomogeneous wave equation

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]G\left(\vec{x},t;\vec{x}',t'\right)$$

$$=-4\pi\delta\left(\vec{x}-\vec{x}'\right)\delta\left(t-t'\right)$$

Will pick out the solution with causal boundary conditions

$$G\left(\vec{x},t;\vec{x}',t'\right)=0 \qquad\qquad t<t'$$

This choice leads automatically to the so-called *Retarded* Green Function

In general

$$G\left(\vec{x},t;\vec{x}',t'\right)=0 \qquad t<t'$$

$$G\left(\vec{x},t;\vec{x}',t'\right)=$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[A\left(\vec{k}\right)e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}+B\left(\vec{k}\right)e^{i\left(\vec{k}\cdot\vec{x}+\omega t\right)}\right]d^3\vec{k} \qquad t>t'$$

because there are two possible signs of the frequency for each value of the wave vector. To solve the homogeneous wave equation it is necessary that

$$\omega\left(\vec{k}\right)=\left|\vec{k}\right|c$$

i.e., there is no dispersion in free space.

Continuity of G implies

$$A\left(\vec{k}\right)e^{-i\omega t'} = -B\left(\vec{k}\right)e^{i\omega t'}$$

Integrate the inhomogeneous equation between $t = t' + \varepsilon$ and $t = t' - \varepsilon$

$$-\frac{1}{c^2}\frac{\partial G\left(\vec{x},t;\vec{x}',t'\right)}{\partial t}\bigg|_{t'+\varepsilon} = -4\pi\delta\left(\vec{x}-\vec{x}'\right)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[-i\omega A\left(\vec{k}\right)e^{i\left(\vec{k}\cdot\vec{x}-\omega t'\right)} + i\omega B\left(\vec{k}\right)e^{i\left(\vec{k}\cdot\vec{x}+\omega t'\right)}\right]\, d^3\vec{k}$$

$$= 4\pi c^2\delta\left(\vec{x}-\vec{x}'\right)$$

$$A\left(\vec{k}\right) = -\frac{c^2}{\left(2\pi\right)^2 i\omega}e^{-i\vec{k}\cdot\vec{x}'}e^{i\omega t'}$$

$$G\left(\vec{x},t;\vec{x}',t'\right)=$$

$$-\frac{c^2}{(2\pi)^2 i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\omega}\left[e^{i\left(\vec{k}\cdot(\vec{x}-\vec{x}')-\omega(t-t')\right)}-e^{i\left(\vec{k}\cdot(\vec{x}-\vec{x}')+\omega(t-t')\right)}\right]d^3\vec{k}$$

$$t>t'$$

$$=\frac{c}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ik\left|\vec{x}-\vec{x}'\right|}}{\left|\vec{x}-\vec{x}'\right|}e^{-i\omega(t-t')}dk-\frac{c}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ik\left|\vec{x}-\vec{x}'\right|}}{\left|\vec{x}-\vec{x}'\right|}e^{+i\omega(t-t')}dk \quad t>t'$$

$$=\frac{\delta\left(\left|\vec{x}-\vec{x}'\right|/c-t+t'\right)}{\left|\vec{x}-\vec{x}'\right|}+0$$

Called retarded because the influence at time *t* is due to the source evaluated at the retarded time

$$t'=t-\left|\vec{x}-\vec{x}'\right|/c$$

Retarded Solutions for Fields

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\phi=-\frac{\rho}{\varepsilon_0}$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{A}=-\mu_0\vec{J}$$

$$\phi(\vec{x},t)=\frac{1}{4\pi\varepsilon_0}\int d^3x'dt'\frac{\rho(\vec{x}',t')}{|\vec{x}-\vec{x}'|}\delta\left(|\vec{x}-\vec{x}'|/c-t+t'\right)$$

$$\vec{A}(\vec{x},t)=\frac{\mu_0}{4\pi}\int d^3x'dt'\frac{\vec{J}(\vec{x}',t')}{|\vec{x}-\vec{x}'|}\delta\left(|\vec{x}-\vec{x}'|/c-t+t'\right)$$

Tip: Leave the delta function in it's integral form to do derivations. Don't have to remember complicated delta-function rules

Retarded Solutions for Fields

$$\phi\left(\vec{x},t\right)=\frac{1}{8\pi^2\varepsilon_0}\int d^3x'dt'd\omega\,\frac{\rho\left(\vec{x}',t'\right)}{\left|\vec{x}-\vec{x}'\right|}e^{i\omega\left[\left|\vec{x}-\vec{x}'\right|/c-\left(t-t'\right)\right]}$$

$$\vec{A}\left(\vec{x},t\right)=\frac{\mu_0}{8\pi^2}\int d^3x'dt'd\omega\,\frac{\vec{J}\left(\vec{x}',t'\right)}{\left|\vec{x}-\vec{x}'\right|}e^{i\omega\left[\left|\vec{x}-\vec{x}'\right|/c-\left(t-t'\right)\right]}$$

Evaluation can be expedited by noting and using the symmetry of the Green function and using relations such as

$$\frac{\partial}{\partial t}f\left(t-t'\right)=-\frac{\partial}{\partial t'}f\left(t-t'\right)$$

$$\frac{\partial}{\partial \vec{x}}f\left(\left|\vec{x}-\vec{x}'\right|\right)=-\frac{\partial}{\partial \vec{x}'}f\left(\left|\vec{x}-\vec{x}'\right|\right)$$

Retarded Solutions for Fields

$$\phi\left(\vec{x},t\right)=\frac{1}{8\pi^2\varepsilon_0}\int d^3x'dt'd\omega\,\frac{\rho\left(\vec{x}',t'\right)}{\left|\vec{x}-\vec{x}'\right|}e^{i\omega\left[\left|\vec{x}-\vec{x}'\right|/c-\left(t-t'\right)\right]}$$

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Evaluation can be expedited by noting and using the symmetry of the Green function and using relations such as

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$$\frac{\partial}{\partial\vec{x}}f\left(\left|\vec{x}-\vec{x}'\right|\right)=-\frac{\partial}{\partial\vec{x}'}f\left(\left|\vec{x}-\vec{x}'\right|\right)$$

Radiation From Relativistic Electrons

From discussion early in the course, in the Lorenz gauge the equation for the potentials is

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\phi=-\frac{\rho}{\varepsilon_0}$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{A}=-\mu_0\vec{J}$$

The solution, using the retarded Green Function is

$$\phi\left(\vec{x},t\right)=\frac{1}{4\pi\varepsilon_0}\int d^3x'dt'\,\frac{\rho\left(\vec{x}',t'\right)}{\left|\vec{x}-\vec{x}'\right|}\delta\left(\left|\vec{x}-\vec{x}'\right|/c-t+t'\right)$$

$$\vec{A}\left(\vec{x},t\right)=\frac{\mu_0}{4\pi}\int d^3x'dt'\,\frac{\vec{J}\left(\vec{x}',t'\right)}{\left|\vec{x}-\vec{x}'\right|}\delta\left(\left|\vec{x}-\vec{x}'\right|/c-t+t'\right)$$

Delta Function Representation

$$\phi(\vec{x},t)=\frac{1}{8\pi^2\varepsilon_0}\int d^3x'dt'd\omega\,\frac{\rho(\vec{x}',t')}{|\vec{x}-\vec{x}'|}e^{i\omega\left[|\vec{x}-\vec{x}'|/c-(t-t')\right]}$$

$$\vec{A}(\vec{x},t)=\frac{\mu_0}{8\pi^2}\int d^3x'dt'd\omega\,\frac{\vec{J}(\vec{x}',t')}{|\vec{x}-\vec{x}'|}e^{i\omega\left[|\vec{x}-\vec{x}'|/c-(t-t')\right]}$$

$$\rho(\vec{x},t)=q\delta^3(\vec{x}-\vec{r}(t)) \qquad \vec{J}(\vec{x},t)=q\vec{\mathrm{v}}(t)\delta^3(\vec{x}-\vec{r}(t))$$

$$\phi(\vec{x},t)=\frac{q}{8\pi^2\varepsilon_0}\int dt'd\omega\,\frac{1}{|\vec{x}-\vec{r}(t')|}e^{i\omega\left[|\vec{x}-\vec{r}(t')|/c-(t-t')\right]}$$

$$\vec{A}(\vec{x},t)=\frac{q\mu_0}{8\pi^2}\int dt'd\omega\,\frac{\vec{\mathrm{v}}(t')}{|\vec{x}-\vec{r}(t')|}e^{i\omega\left[|\vec{x}-\vec{r}(t')|/c-(t-t')\right]}$$

Lienard-Weichert Potentials

$$\phi(\vec{x},t)=\frac{q}{4\pi\varepsilon_0}\int dt'\,\frac{\delta\left(\left|\vec{x}-\vec{r}(t')\right|/c-(t-t')\right)}{\left|\vec{x}-\vec{r}(t')\right|}$$

$$\vec{A}(\vec{x},t)=\frac{q\mu_0}{4\pi}\int dt'\,\frac{\vec{\mathrm{v}}(t')\,\delta\left(\left|\vec{x}-\vec{r}(t')\right|/c-(t-t')\right)}{\left|\vec{x}-\vec{r}(t')\right|}$$

$$\phi(\vec{x},t)=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\left|\vec{x}-\vec{r}(t')\right|\left(1-\hat{n}\cdot\vec{\beta}(t')\right)}\right)_{ret}$$

$$\vec{A}(\vec{x},t)=\frac{q\mu_0}{4\pi}\left(\frac{\vec{\mathrm{v}}(t')}{\left|\vec{x}-\vec{r}(t')\right|\left(1-\hat{n}\cdot\vec{\beta}(t')\right)}\right)_{ret}$$

EM Field Radiated

$$\phi(\vec{x},t)=\frac{q}{8\pi^2\varepsilon_0}\int dt'd\omega\,\frac{1}{\left|\vec{x}-\vec{r}(t')\right|}e^{i\omega\left[\left|\vec{x}-\vec{r}(t')\right|/c-(t-t')\right]}$$

$$\vec{A}(\vec{x},t)=\frac{q\mu_0}{8\pi^2}\int dt'd\omega\,\frac{\vec{\mathrm{v}}(t')}{\left|\vec{x}-\vec{r}(t')\right|}e^{i\omega\left[\left|\vec{x}-\vec{r}(t')\right|/c-(t-t')\right]}$$

$$\vec{E}=-\vec{\nabla}\phi-\frac{\partial\vec{A}}{\partial t}\qquad\qquad \vec{B}=\vec{\nabla}\times\vec{A}$$

$$\vec{E}=\frac{q}{4\pi\varepsilon_0}\left[\frac{\hat{n}-\vec{\beta}}{\gamma^2\left(1-\hat{n}\cdot\vec{\beta}\right)^3R^2}\right]_{ret}+\frac{q}{4\pi\varepsilon_0 c}\left[\frac{\hat{n}\times\left\{\left(\hat{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right\}}{\left(1-\hat{n}\cdot\vec{\beta}\right)^3R}\right]_{ret}$$

$$\vec{B}=\hat{n}\times\vec{E}/c$$

$$\vec{\nabla}\frac{1}{\left|\vec{x}-\vec{r}\left(t'\right)\right|}=-\frac{\hat{n}}{\left|\vec{x}-\vec{r}\left(t'\right)\right|^{2}}\qquad \vec{\nabla}\left|\vec{x}-\vec{r}\left(t'\right)\right|=\hat{n}$$

$$\frac{d}{dt'}\frac{1}{\left|\vec{x}-\vec{r}\left(t'\right)\right|}=\frac{\hat{n}\cdot\vec{\beta}c}{\left|\vec{x}-\vec{r}\left(t'\right)\right|^{2}}\qquad \frac{d\hat{n}}{dt'}=\frac{-d\vec{r}/dt'+\hat{n}\left(\hat{n}\cdot d\vec{r}/dt'\right)}{\left|\vec{x}-\vec{r}\left(t'\right)\right|}\qquad\cdots$$

$$-\nabla\phi\left(\vec{x},t\right)=\frac{q}{8\pi^{2}\varepsilon_{0}}\int dt'd\omega\frac{\hat{n}\left(1-i\omega\left|\vec{x}-\vec{r}\left(t'\right)\right|/c\right)}{\left|\vec{x}-\vec{r}\left(t'\right)\right|^{2}}e^{i\omega\left[\left|\vec{x}-\vec{r}\left(t'\right)\right|/c-\left(t-t'\right)\right]}$$

$$-\frac{\partial}{\partial t}\vec{A}\left(\vec{x},t\right)=\frac{q\mu_{0}}{8\pi^{2}}\int dt'd\omega\frac{\vec{\mathrm{v}}\left(t'\right)i\omega}{\left|\vec{x}-\vec{r}\left(t'\right)\right|}e^{i\omega\left[\left|\vec{x}-\vec{r}\left(t'\right)\right|/c-\left(t-t'\right)\right]}$$

$$\frac{d}{dt'}e^{i\omega\left[\left|\vec{x}-\vec{r}\left(t'\right)\right|/c-\left(t-t'\right)\right]}=i\omega\left(1-\vec{\beta}\left(t'\right)\cdot\hat{n}\left(t'\right)\right)e^{i\omega\left[\left|\vec{x}-\vec{r}\left(t'\right)\right|/c-\left(t-t'\right)\right]}$$

$$\vec{E}(\vec{x},t)=\frac{q}{8\pi^2\varepsilon_0}\int dt'd\omega e^{i\omega\left[\left|\vec{x}-\vec{r}(t')\right|/c-(t-t')\right]}\left[\frac{\hat{n}}{\left|\vec{x}-\vec{r}(t')\right|^2}+\frac{i\omega\left(\vec{\beta}-\hat{n}\right)}{c\left|\vec{x}-\vec{r}(t')\right|}\right]$$

integrate by parts to get final result

$$\vec{E}(\vec{x},t)_{vel}=\frac{q}{8\pi^2\varepsilon_0}\int dt'd\omega\frac{e^{i\omega\left[\left|\vec{x}-\vec{r}(t')\right|/c-(t-t')\right]}}{\left(1-\vec{\beta}\cdot\hat{n}\right)^2\left|\vec{x}-\vec{r}(t')\right|^2}$$

$$\times\left[\begin{array}{l}\hat{n}\left(\begin{array}{l}1-2\vec{\beta}\cdot\hat{n}+\left(\vec{\beta}\cdot\hat{n}\right)^2+\vec{\beta}\cdot\hat{n}-\left(\vec{\beta}\cdot\hat{n}\right)^2+\vec{\beta}\cdot\hat{n}-\left(\vec{\beta}\cdot\hat{n}\right)^2\\-\beta^2+\left(\vec{\beta}\cdot\hat{n}\right)^2\end{array}\right)\\-\vec{\beta}\left(1-\vec{\beta}\cdot\hat{n}+\vec{\beta}\cdot\hat{n}-\left(\vec{\beta}\cdot\hat{n}\right)^2-\beta^2+\left(\vec{\beta}\cdot\hat{n}\right)^2\right)\end{array}\right]$$

$$\vec{E}\left(\vec{x},t\right)_{acc} = \frac{q}{8\pi^2\varepsilon_0 c}\int dt'd\omega \frac{e^{i\omega\left[\left|\vec{x}-\vec{r}\left(t'\right)\right|/c-\left(t-t'\right)\right]}}{\left(1-\vec{\beta}\cdot\hat{n}\right)^2\left|\vec{x}-\vec{r}\left(t'\right)\right|}$$

$$\times\begin{bmatrix}\hat{n}\left(\dot{\vec{\beta}}\cdot\hat{n}\right) \\ -\dot{\vec{\beta}}\left(1-\vec{\beta}\cdot\hat{n}\right)-\vec{\beta}\left(\dot{\vec{\beta}}\cdot\hat{n}\right)\end{bmatrix}$$

$$= \frac{q}{8\pi^2\varepsilon_0 c}\int dt'd\omega \frac{e^{i\omega\left[\left|\vec{x}-\vec{r}\left(t'\right)\right|/c-\left(t-t'\right)\right]}}{\left(1-\vec{\beta}\cdot\hat{n}\right)^2\left|\vec{x}-\vec{r}\left(t'\right)\right|}\left[\hat{n}\times\left\{\left(\hat{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right\}\right]$$

Larmor's Formula

For small velocities can neglect retardation

$$\vec{E}\left(\vec{x},t\right)_{acc} = \frac{q}{4\pi\varepsilon_0 c}\left[\hat{n}\times\left\{\hat{n}\times\dot{\vec{\beta}}\right\}\right]/R$$

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0\mu_0 c^3}\left|\left[\hat{n}\times\left\{\hat{n}\times\dot{\vec{\beta}}\right\}\right]\right|^2$$

$$= \frac{q^2}{16\pi^2\varepsilon_0 c^3}\left|\dot{\vec{\mathrm{v}}}\right|^2 \sin^2\theta$$

$$P = \frac{q^2}{6\pi\varepsilon_0 c^3}\left|\dot{\vec{\mathrm{v}}}\right|^2$$

Relativistic Peaking

In far field after short acceleration

$$\frac{dP(t')}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0 c} \frac{\left| \hat{n} \times \left\{ \left(\hat{n} - \vec{\beta} \right) \times \dot{\vec{\beta}} \right\} \right|^2}{\left(1 - \hat{n} \cdot \vec{\beta} \right)^5}$$

$$\frac{dP(t')}{d\Omega} = \frac{q^2 \dot{\beta}^2}{16\pi^2\varepsilon_0 c} \frac{\sin^2\theta}{\left(1 - \beta\cos\theta\right)^5}$$

$$\theta_{\max} \to \frac{1}{2\gamma}$$

For circular motions

$$\frac{dP(t')}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0 c} \frac{\dot{\beta}^2}{\left(1 - \beta\cos\theta\right)^3} \left[1 - \frac{\sin^2\theta\cos^2\varphi}{\gamma^2\left(1 - \beta\cos\theta\right)^2} \right]$$

Spectrum Radiated by Motion

$$\frac{dE}{d\Omega} = \int_{-\infty}^{\infty} \frac{dP}{d\Omega} dt = \int_{-\infty}^{\infty} \vec{E} \times \vec{H} \cdot \hat{n} R^2 dt = \frac{1}{c\mu_0} \int_{-\infty}^{\infty} \left(\vec{E} \cdot \vec{E} \right) R^2 dt =$$

$$\frac{1}{c\mu_0} \left(\frac{q}{8\pi^2 \varepsilon_0 c} \right)^2 \int_{-\infty}^{\infty} \int \int \int \int \left[\frac{\hat{n} \times \left\{ \left(\hat{n} - \vec{\beta} \right) \times \dot{\vec{\beta}} \right\}}{\left(1 - \hat{n} \cdot \vec{\beta} \right)^2} (t') \right] \cdot \left[\frac{\hat{n} \times \left\{ \left(\hat{n} - \vec{\beta} \right) \times \dot{\vec{\beta}} \right\}}{\left(1 - \hat{n} \cdot \vec{\beta} \right)^2} (t'') \right]$$

$$\times e^{i\omega \left[R\sqrt{1 - 2\hat{n}\cdot\vec{r}(t')/R + \left(\hat{n}\cdot\vec{r}(t') \right)^2 / R^2} / c - t + t' \right]} e^{i\omega' \left[\sqrt{1 - 2\hat{n}\cdot\vec{r}(t'')/R + \left(\hat{n}\cdot\vec{r}(t'') \right)^2 / R^2} / c - t + t'' \right]} dt' d\omega dt'' d\omega' dt =$$

clearing the unprimed time integral and omega prime integral with delta represntation

$$\frac{2\pi}{c\mu_0} \left(\frac{q}{8\pi^2 \varepsilon_0 c} \right)^2 \int_{-\infty}^{\infty} \int \int \left[\frac{\hat{n} \times \left\{ \left(\hat{n} - \vec{\beta} \right) \times \dot{\vec{\beta}} \right\}}{\left(1 - \hat{n} \cdot \vec{\beta} \right)^2} (t') \right] \cdot \left[\frac{\hat{n} \times \left\{ \left(\hat{n} - \vec{\beta} \right) \times \dot{\vec{\beta}} \right\}}{\left(1 - \hat{n} \cdot \vec{\beta} \right)^2} (t'') \right]$$

$$\times e^{i\omega \left[-\hat{n}\cdot\vec{r}(t')/c - t + t' \right]} e^{-i\omega \left[-\hat{n}\cdot\vec{r}(t'')/c - t + t'' \right]} dt' dt'' d\omega$$

$$\frac{d^2E}{d\omega d\Omega} = \frac{q^2}{32\pi^3\varepsilon_0 c}\left|\int_{-\infty}^{\infty}\frac{\hat{n}\times\left\{\left(\hat{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right\}}{\left(1-\hat{n}\cdot\vec{\beta}\right)^2}e^{i\omega\left[-\hat{n}\cdot\vec{r}(t')/c-t+t'\right]}dt'\right|^2$$

$$\frac{d^2E}{d\omega d\Omega} = \frac{q^2\omega^2}{32\pi^3\varepsilon_0 c}\left|\int_{-\infty}^{\infty}\hat{n}\times\left(\hat{n}\times\vec{\beta}\right)e^{i\omega\left[t'-\hat{n}\cdot\vec{r}(t')/c\right]}dt'\right|^2$$

Factor of two difference from Jackson because he combines positive frequency and negative frequency contributions in one positive frequency integral. I don't like because Parseval's formula, etc. don't work! I've written papers about performing this calculation in new regimes of high intensity pulsed lasers.