Betatron Motion with Coupling of Horizontal and Vertical Degrees of Freedom – Part II

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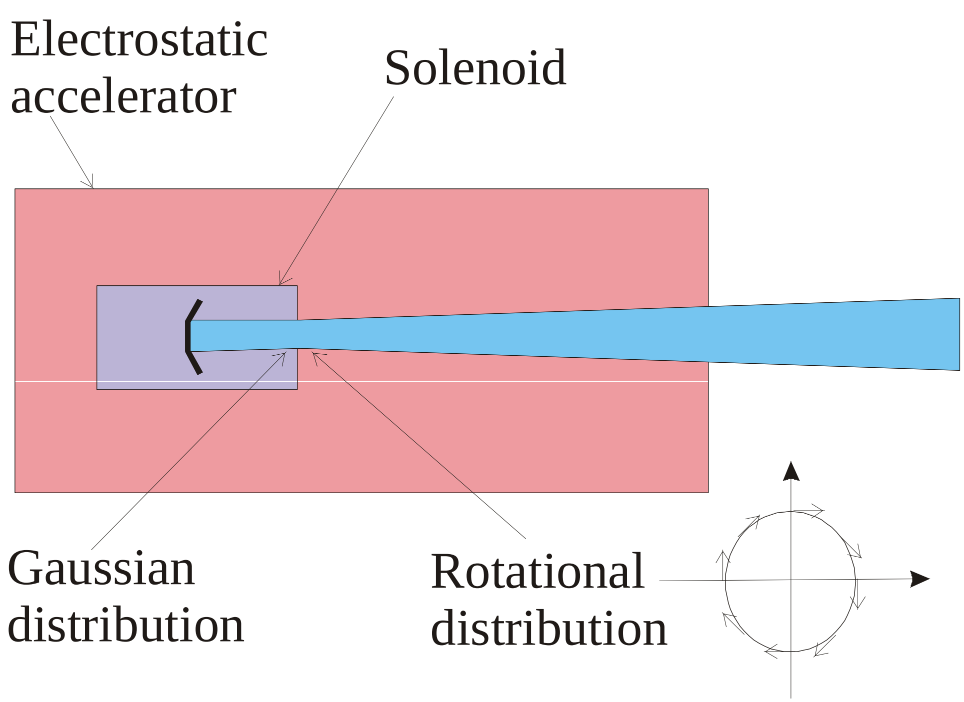
Outline

- Practical Examples:
  - Spin Rotator for Figure-8 Collider ring
  - Vertex-to-plane adapter for electron cooling (Fermilab)
  - Ionization cooling channel for Neutrino Factory and Muon Collider
  - Generalized vertex-to-plane transformer insert

    - V. Lebedev, A. Bogacz, 'Betatron Motion with Coupling of Horizontal and Vertical Degrees of Freedom', 2000, http://dx.doi.org/10.1088/1748-0221/5/10/P10010

Axisymmetric rotational distribution – Twiss functions

❖Fermilab electron cooling



The electron beam distribution is axially symmetric, and uncoupled at the cathode:

$$\mathbf{\Xi}_B = \frac{1}{\varepsilon_T}\begin{bmatrix} \gamma_0 & \alpha_0 & 0 & 0 \\ \alpha_0 & \beta_0 & 0 & 0 \\ 0 & 0 & \gamma_0 & \alpha_0 \\ 0 & 0 & \alpha_0 & \beta_0 \end{bmatrix}$$

where $\varepsilon_T = r_c\sqrt{mkT_c}\,/\,P_0$ is the thermal emittance of the beam

Axisymmetric rotational distribution – Twiss functions

- At the exit of the solenoid the electron beam distribution is still axially symmetric

$$\boldsymbol{\Xi}_{in} = \boldsymbol{\Phi}^T \, \boldsymbol{\Xi}_B \, \boldsymbol{\Phi} = \frac{1}{\varepsilon_T}\begin{bmatrix} \gamma_0 + \Phi^2\beta_0 & \alpha_0 & 0 & -\Phi\beta_0 \\ \alpha_0 & \beta_0 & \Phi\beta_0 & 0 \\ 0 & \Phi\beta_0 & \gamma_0 + \Phi^2\beta_0 & \alpha_0 \\ -\Phi\beta_0 & 0 & \alpha_0 & \beta_0 \end{bmatrix}$$

where

$$\boldsymbol{\Phi} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \Phi & 0 \\ 0 & 0 & 1 & 0 \\ -\Phi & 0 & 0 & 1 \end{bmatrix}$$

  - $\Phi = eB/2P_0c$ is the rotational focusing strength of the solenoid edge
  - B is the solenoid magnetic field.

Axisymmetric rotational distribution – Twiss functions

- The eigen-vectors of the rotational distribution:

$$\hat{\mathbf{v}}_1 = \begin{bmatrix} \sqrt{\beta} \\ -\dfrac{i+2\alpha}{2\sqrt{\beta}} \\ i\sqrt{\beta} \\ -i\dfrac{i+2\alpha}{2\sqrt{\beta}} \end{bmatrix} \quad , \quad \hat{\mathbf{v}}_2 = \begin{bmatrix} i\sqrt{\beta} \\ -i\dfrac{i+2\alpha}{2\sqrt{\beta}} \\ \sqrt{\beta} \\ -\dfrac{i+2\alpha}{2\sqrt{\beta}} \end{bmatrix}$$

  - It corresponds to *u* = 1/2, $\nu_1 = \nu_2 = \pi / 2$

- Then, the matrix $\hat{\mathbf{V}}$ is

$$\hat{\mathbf{V}} = \begin{bmatrix} \sqrt{\beta} & 0 & 0 & -\sqrt{\beta} \\ -\dfrac{\alpha}{\sqrt{\beta}} & \dfrac{1}{2\sqrt{\beta}} & \dfrac{1}{2\sqrt{\beta}} & \dfrac{\alpha}{\sqrt{\beta}} \\ 0 & -\sqrt{\beta} & \sqrt{\beta} & 0 \\ \dfrac{1}{2\sqrt{\beta}} & \dfrac{\alpha}{\sqrt{\beta}} & -\dfrac{\alpha}{\sqrt{\beta}} & \dfrac{1}{2\sqrt{\beta}} \end{bmatrix}$$

Axisymmetric rotational distribution – Twiss functions

- Comparing left and right hand sides of the equation

$$\hat{\boldsymbol{\Xi}}_{in} = \mathbf{U}\hat{\mathbf{V}}\begin{bmatrix} 1/\varepsilon_1 & 0 & 0 & 0 \\ 0 & 1/\varepsilon_1 & 0 & 0 \\ 0 & 0 & 1/\varepsilon_2 & 0 \\ 0 & 0 & 0 & 1/\varepsilon_2 \end{bmatrix}\hat{\mathbf{V}}^T\mathbf{U}^T$$

- One obtains

$$\beta = \frac{\beta_0}{2\sqrt{1+\Phi^2\beta_0^{\ 2}}} \quad ,$$

$$\alpha = \frac{\alpha_0}{2\sqrt{1+\Phi^2\beta_0^{\ 2}}} \quad ,$$

$$\varepsilon_1 = \frac{\varepsilon_T}{\sqrt{1+\Phi^2\beta_0^{\ 2}} - \Phi\beta_0} \xrightarrow{\Phi\beta_0 >> 1} 2\Phi\beta_0\varepsilon_T \quad ,$$

$$\varepsilon_2 = \frac{\varepsilon_T}{\sqrt{1+\Phi^2\beta_0^{\ 2}} + \Phi\beta_0} \xrightarrow{\Phi\beta_0 >> 1} \frac{\varepsilon_T}{2\Phi\beta_0} \quad .$$

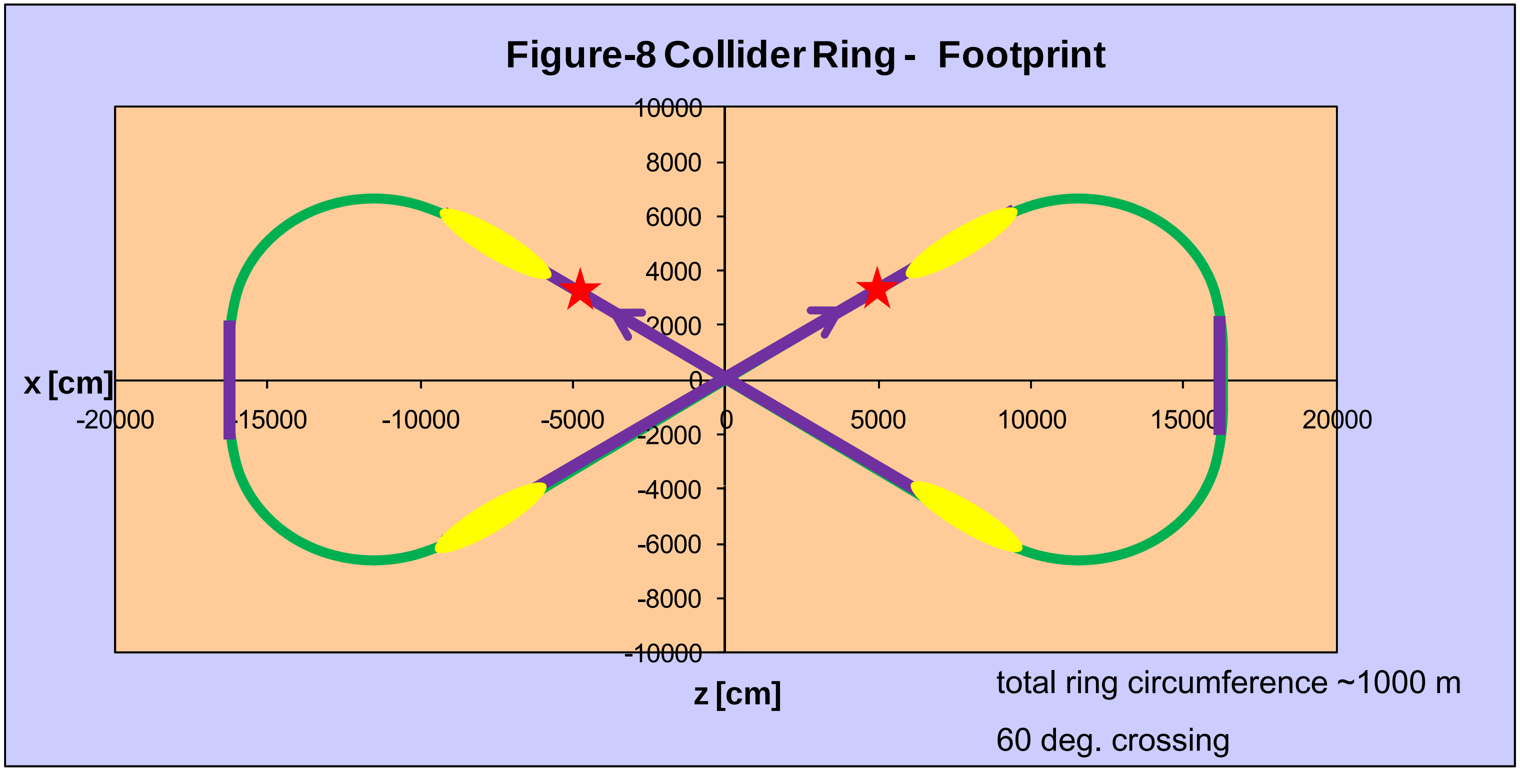
- 4D-emmitance conservation:

$$\varepsilon_1\varepsilon_2 = \varepsilon_T^{\ 2}$$

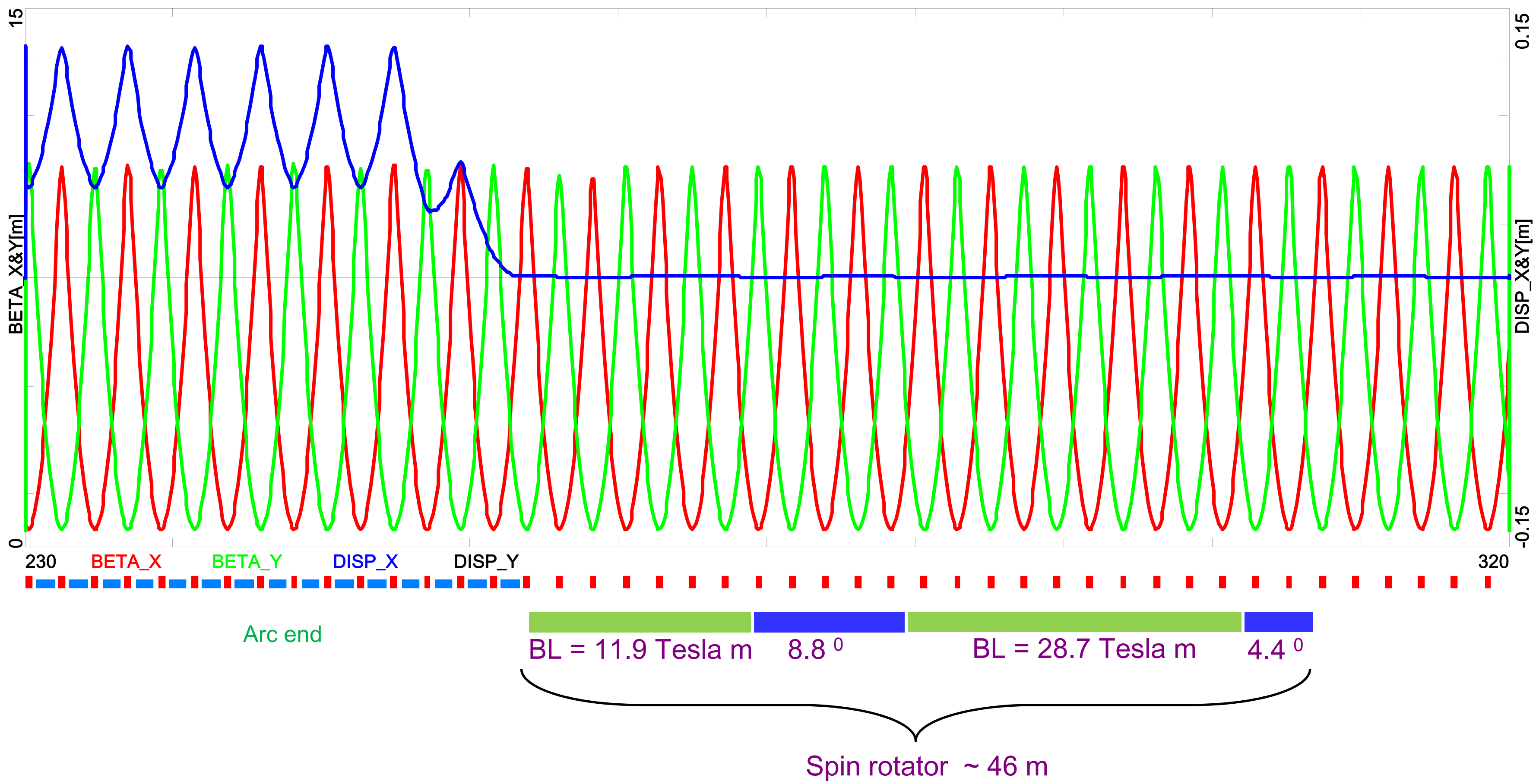
- Rotational emittance estimate

$$\varepsilon_{rot} = r\theta = r(r\Phi) = r^2\Phi = (\varepsilon_T\beta_0)\Phi$$

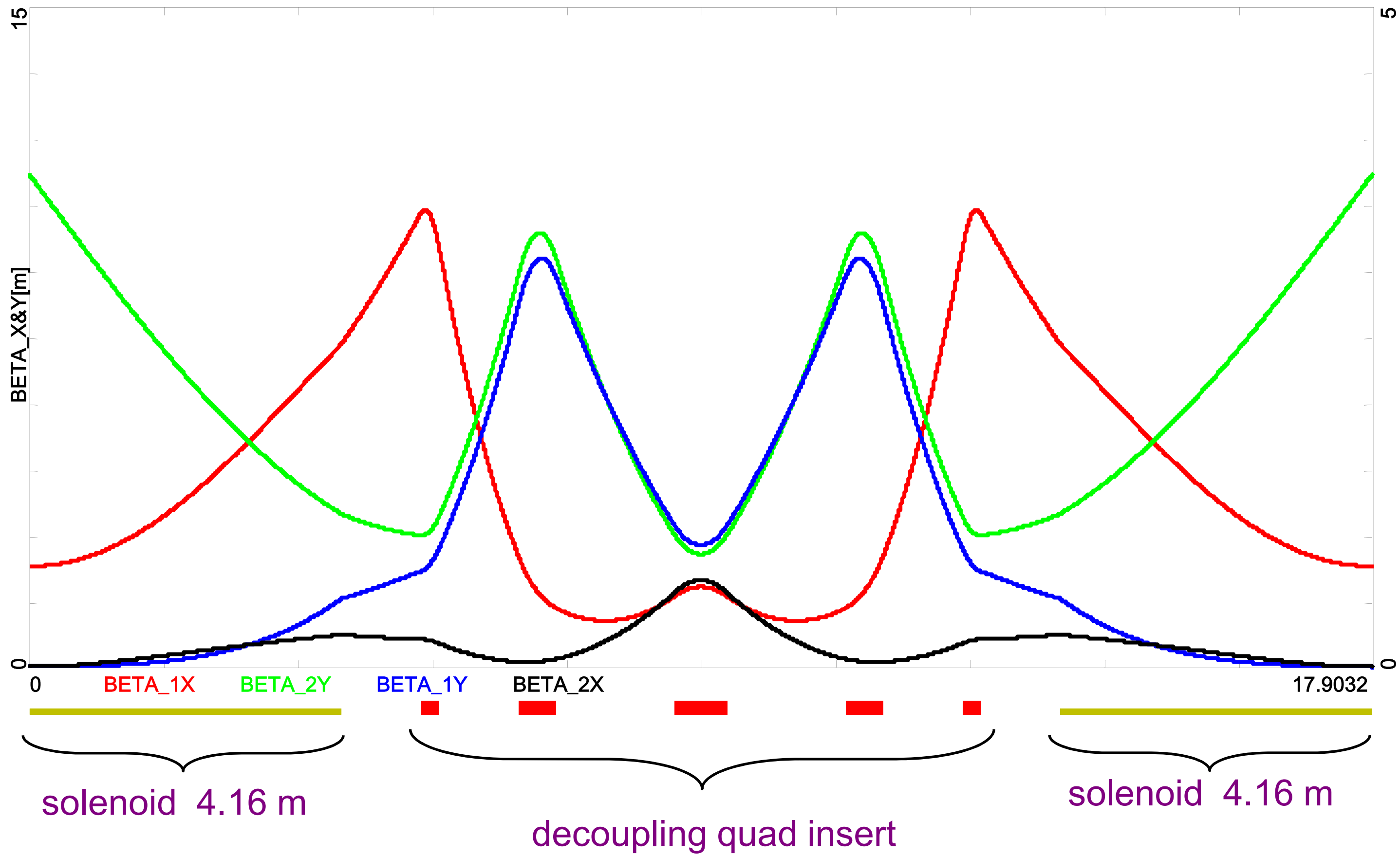
Spin rotators for Figure-8 Collider Ring



Spin Rotator – Ingredients…

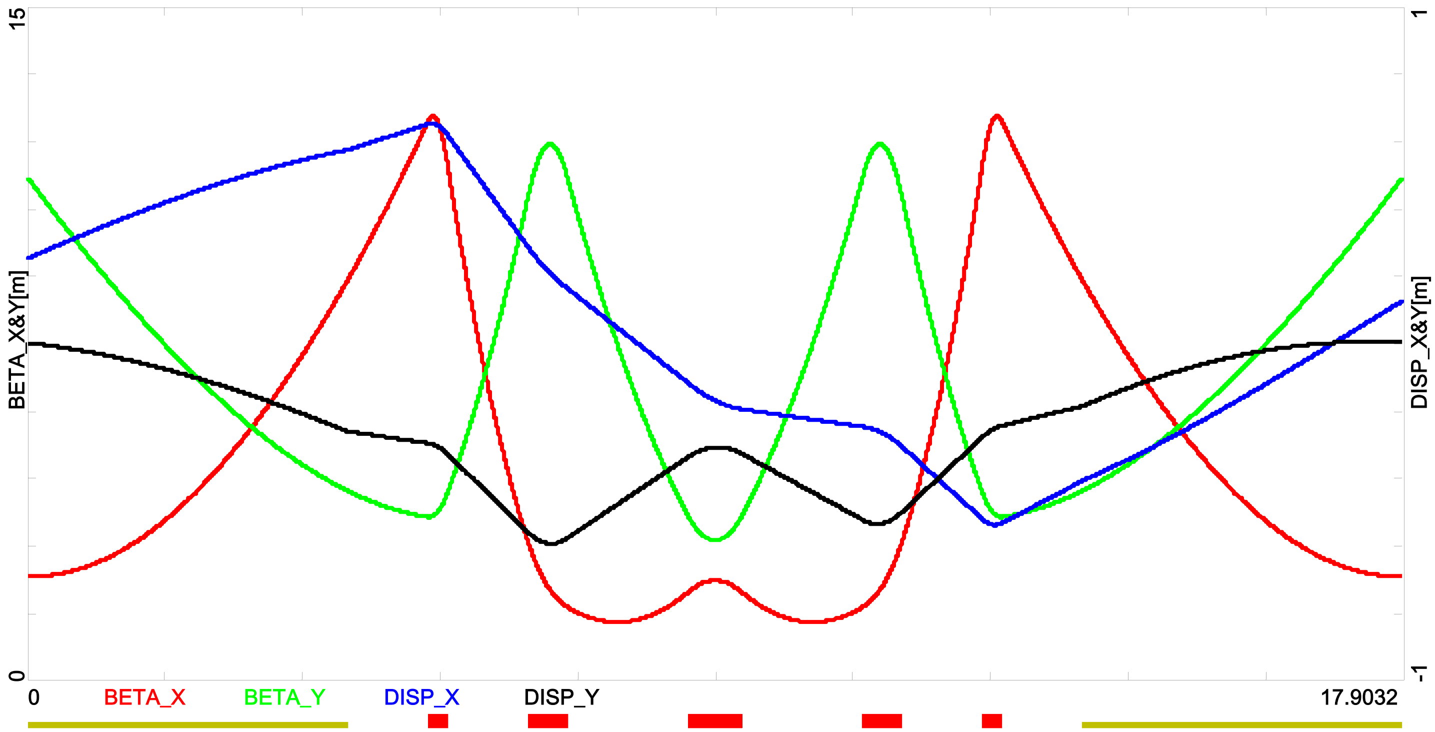


Locally decoupled solenoid pair (Hisham Sayed)



$$M = \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix}$$

Locally decoupled solenoid pair (Hisham Sayed)



solenoid 4.16 m

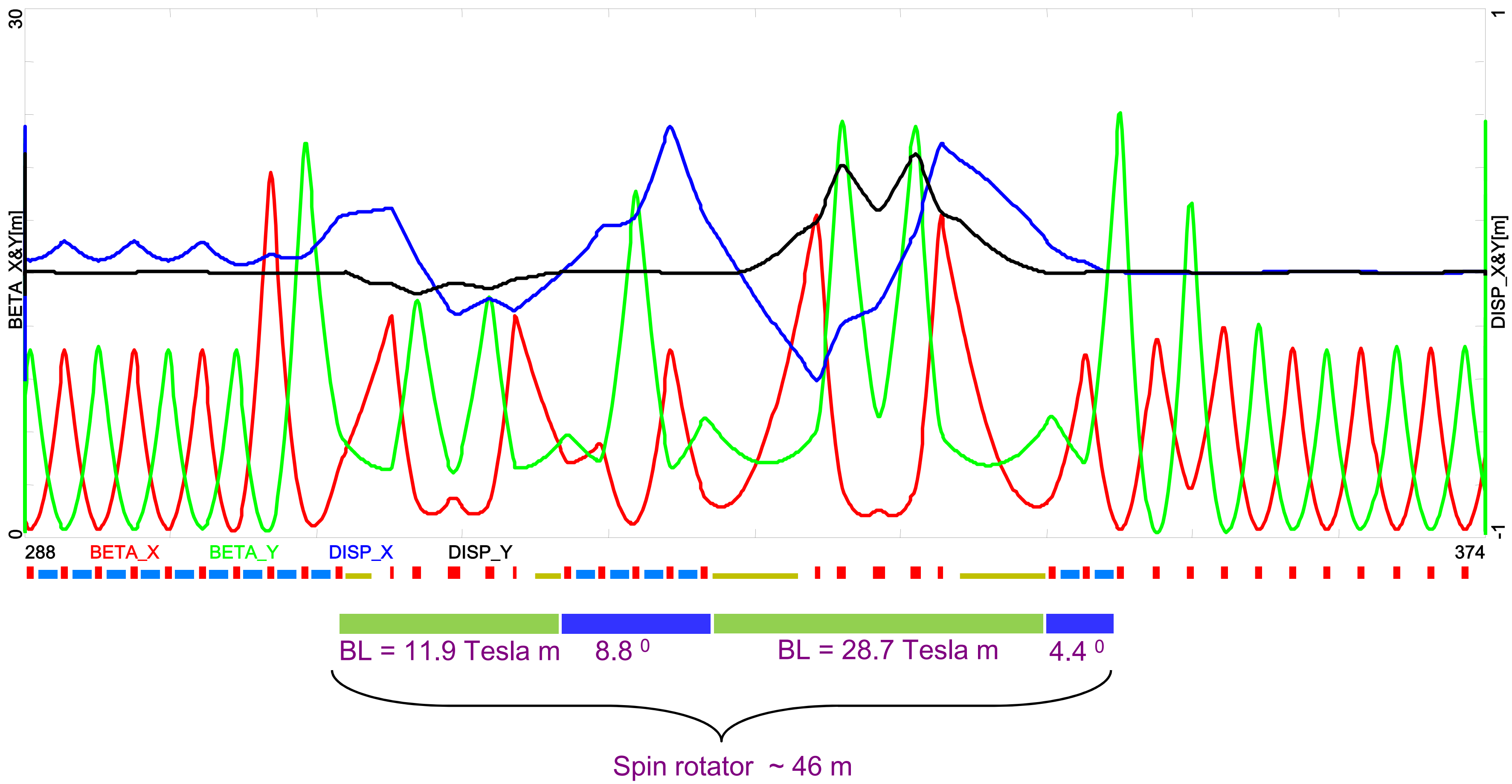
decoupling quad insert

solenoid 4.16 m

$$M = \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix}$$

Universal Spin Rotator – Optics

5 GeV



Ionization Cooling in an Axially Symmetric Channel

- A single-particle phase-space trajectory along the beam orbit can be expressed as:

$$\hat{\mathbf{x}}(s) = \mathrm{Re}\left(\sqrt{\varepsilon_1}\,\hat{\mathbf{v}}_1(s)e^{-i(\psi_1+\mu_1(s))} + \sqrt{\varepsilon_2}\,\hat{\mathbf{v}}_2(s)e^{-i(\psi_2+\mu_2(s))}\right) \quad ,$$

- One can rewrite the above equations in the following compact form

$$\boxed{\hat{\mathbf{x}}(s) = \hat{\mathbf{V}}(s)\mathbf{a}(s)}$$

where

$$\hat{\mathbf{V}}(s) = \left[\hat{\mathbf{v}}_1{}'(s), -\hat{\mathbf{v}}_1{}''(s), \hat{\mathbf{v}}_2{}'(s), -\hat{\mathbf{v}}_2{}''(s)\right] \qquad \mathbf{a}(s) = \begin{bmatrix} \sqrt{\varepsilon_1}\cos(\psi_1+\mu_1(s)) \\ \sqrt{\varepsilon_1}\sin(\psi_1+\mu_1(s)) \\ \sqrt{\varepsilon_2}\cos(\psi_2+\mu_2(s)) \\ \sqrt{\varepsilon_2}\sin(\psi_2+\mu_2(s)) \end{bmatrix}$$

Ionization Cooling in an Axially Symmetric Channel

- In the case of axially symmetric focusing the eigen-vectors reduce to

$$\hat{\mathbf{v}}_1 = \begin{bmatrix} \sqrt{\beta} \\ -\dfrac{i+2\alpha}{2\sqrt{\beta}} \\ i\sqrt{\beta} \\ -i\dfrac{i+2\alpha}{2\sqrt{\beta}} \end{bmatrix} \quad , \qquad \hat{\mathbf{v}}_2 = \begin{bmatrix} i\sqrt{\beta} \\ -i\dfrac{i+2\alpha}{2\sqrt{\beta}} \\ \sqrt{\beta} \\ -\dfrac{i+2\alpha}{2\sqrt{\beta}} \end{bmatrix}$$

$$\hat{\mathbf{V}} = \begin{bmatrix} \sqrt{\beta} & 0 & 0 & -\sqrt{\beta} \\ -\dfrac{\alpha}{\sqrt{\beta}} & \dfrac{1}{2\sqrt{\beta}} & \dfrac{1}{2\sqrt{\beta}} & \dfrac{\alpha}{\sqrt{\beta}} \\ 0 & -\sqrt{\beta} & \sqrt{\beta} & 0 \\ \dfrac{1}{2\sqrt{\beta}} & \dfrac{\alpha}{\sqrt{\beta}} & -\dfrac{\alpha}{\sqrt{\beta}} & \dfrac{1}{2\sqrt{\beta}} \end{bmatrix}$$

- here we used that $u = 1/2$, $\nu_1 = \nu_2 = \pi/2$

Ionization Cooling in an Axially Symmetric Channel

❖Cooling Description

- Ionization cooling due to energy loss in a thin absorber can be described as:

$$\Delta\theta_\perp = -\theta_\perp \frac{\Delta p}{p} \equiv -\theta_\perp \delta \ ,$$

  - here the longitudinal energy restoration by immediate re-acceleration is assumed

- Using canonical variables the above cooling equation can be written as:

$$\hat{\mathbf{x}}_{out} = \mathbf{R}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-\delta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1-\delta \end{bmatrix}\mathbf{R}^{-1}\hat{\mathbf{x}}_{in}$$

Ionization Cooling in an Axially Symmetric Channel

- Employing amplitude vector representation: $\hat{\mathbf{x}}(s)=\hat{\mathbf{V}}(s)\mathbf{a}(s)$, one can rewrite the cooling equation as:

$$\hat{\mathbf{V}}\mathbf{a}_{out}=\mathbf{R}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1-\delta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1-\delta\end{bmatrix}\mathbf{R}^{-1}\hat{\mathbf{V}}\mathbf{a}_{in}$$

and finally

$$\mathbf{a}_{out}=\hat{\mathbf{V}}^{-1}\mathbf{R}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1-\delta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1-\delta\end{bmatrix}\mathbf{R}^{-1}\hat{\mathbf{V}}\mathbf{a}_{in}$$

Ionization Cooling in an Axially Symmetric Channel

- Carrying out the above calculation explicitly one obtains:

$$\mathbf{a}_{out} = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \delta \begin{bmatrix} \frac{1-R\beta}{2} & \alpha & -\alpha & \frac{1+R\beta}{2} \\ -\alpha & \frac{1-R\beta}{2} & \frac{1+R\beta}{2} & \alpha \\ -\alpha & \frac{1-R\beta}{2} & \frac{1+R\beta}{2} & \alpha \\ \frac{1-R\beta}{2} & \alpha & -\alpha & \frac{1+R\beta}{2} \end{bmatrix} \right\} \mathbf{a}_{in}$$

- 2D emittances after cooling are given by the following formula:

$$\varepsilon_1' \equiv a_{out_1}{}^2 + a_{out_2}{}^2 = \varepsilon_1\left[1-(1-\beta R)\delta\right] + \sqrt{\varepsilon_1\varepsilon_2}\left[2\alpha\cos\phi - (1+\beta R)\sin\phi\right]\delta + O(\delta^2)$$

$$\varepsilon_2' \equiv a_{out_3}{}^2 + a_{out_4}{}^2 = \varepsilon_2\left[1-(1+\beta R)\delta\right] + \sqrt{\varepsilon_1\varepsilon_2}\left[2\alpha\cos\phi - (1-\beta R)\sin\phi\right]\delta + O(\delta^2)$$

where $\phi = \mu_1 + \mu_2 + \psi_1 + \psi_2$

Ionization Cooling in an Axially Symmetric Channel

❖ Two alternative descriptions of cooling

- After passing through a thin absorber one computes
  - a new 4D phase space
  - new partial emittances
  - new beta-functions
- Or one can compute everything relative to the unperturbed beta-functions
  - Seems like more convenient approach, although partial emittances (actions) depend on betatron phases

Ionization Cooling in an Axially Symmetric Channel

- If cooling effect of one absorber is sufficiently small one can perform averaging over betatron phases. That yields

$$\begin{aligned}\Delta\varepsilon_1 &\approx -\varepsilon_1(1-\beta R)\delta \\ \Delta\varepsilon_2 &\approx -\varepsilon_2(1+\beta R)\delta\end{aligned} \quad \Rightarrow \quad \boxed{\begin{aligned}\frac{1}{\varepsilon_1}\frac{d\varepsilon_1}{ds} &\approx -\frac{1-\beta R}{p_0}\frac{dp}{ds} \\ \frac{1}{\varepsilon_2}\frac{d\varepsilon_2}{ds} &\approx -\frac{1+\beta R}{p_0}\frac{dp}{ds}\end{aligned}}$$

Ionization Cooling in an Axially Symmetric Channel

- Canonical momentum of a single particle

$$\mathbf{M} = xp_y - yp_x = \hat{\mathbf{x}}^T \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \hat{\mathbf{x}} = \left(\hat{\mathbf{V}}\mathbf{a}\right)^T \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \hat{\mathbf{V}}\mathbf{a} = \frac{\varepsilon_1 - \varepsilon_2}{2}$$

Ionization Cooling in an Axially Symmetric Channel

- Second order moments of the Gaussian distribution

(Note that for a single particle - $\varepsilon_{rms} = \varepsilon / 2$ and we use rms. emittances below)

$$\langle x^2 \rangle = \langle y^2 \rangle = \beta(\varepsilon_1 + \varepsilon_2) \quad ,$$

$$\langle x p_x \rangle = \langle y p_y \rangle = -\alpha(\varepsilon_1 + \varepsilon_2) \quad ,$$

$$\langle p_x{}^2 \rangle = \langle p_y{}^2 \rangle = \frac{1 + 4\alpha^2}{4\beta}(\varepsilon_1 + \varepsilon_2) \quad ,$$

$$\langle x p_y \rangle = -\langle y p_x \rangle = \frac{\varepsilon_1 - \varepsilon_2}{2} \quad , \quad \Rightarrow \quad \langle M \rangle = \varepsilon_1 - \varepsilon_2$$

$$\langle xy \rangle = \langle p_x p_y \rangle = 0$$

Ionization Cooling in an Axially Symmetric Channel

❖ Beta-function for Particle Motion with Axial-symmetric Solenoidal Focusing

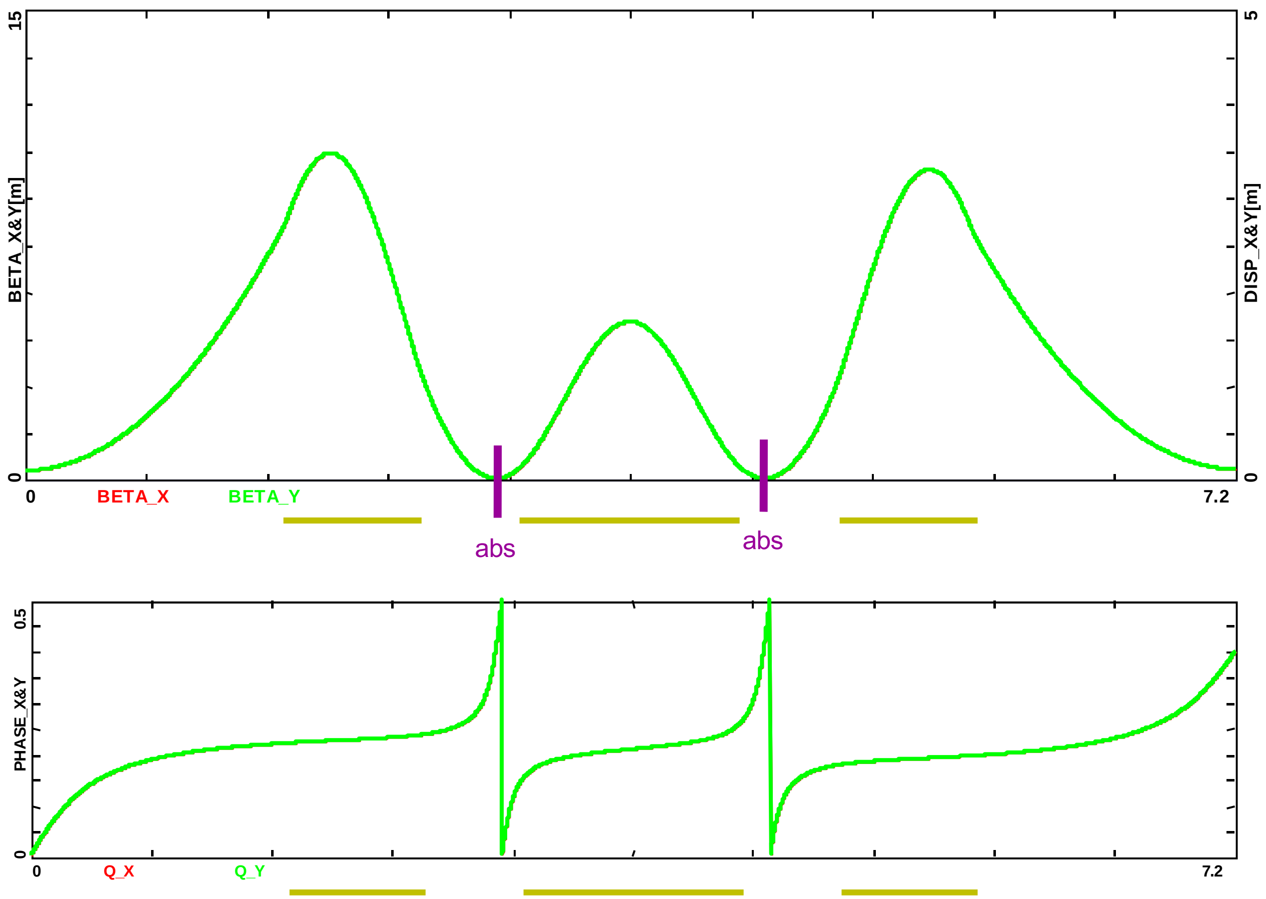
♦ Equation for the square root of the beta-function is similar to the equation for Floque-function in the case of uncoupled motion:

$$\frac{d^2\sqrt{\beta}}{ds^2}+\frac{R^2}{4}\sqrt{\beta}-\frac{1}{4\left(\sqrt{\beta}\right)^3}=0 \quad .$$

♠ The standard recipe determines the alpha-function:

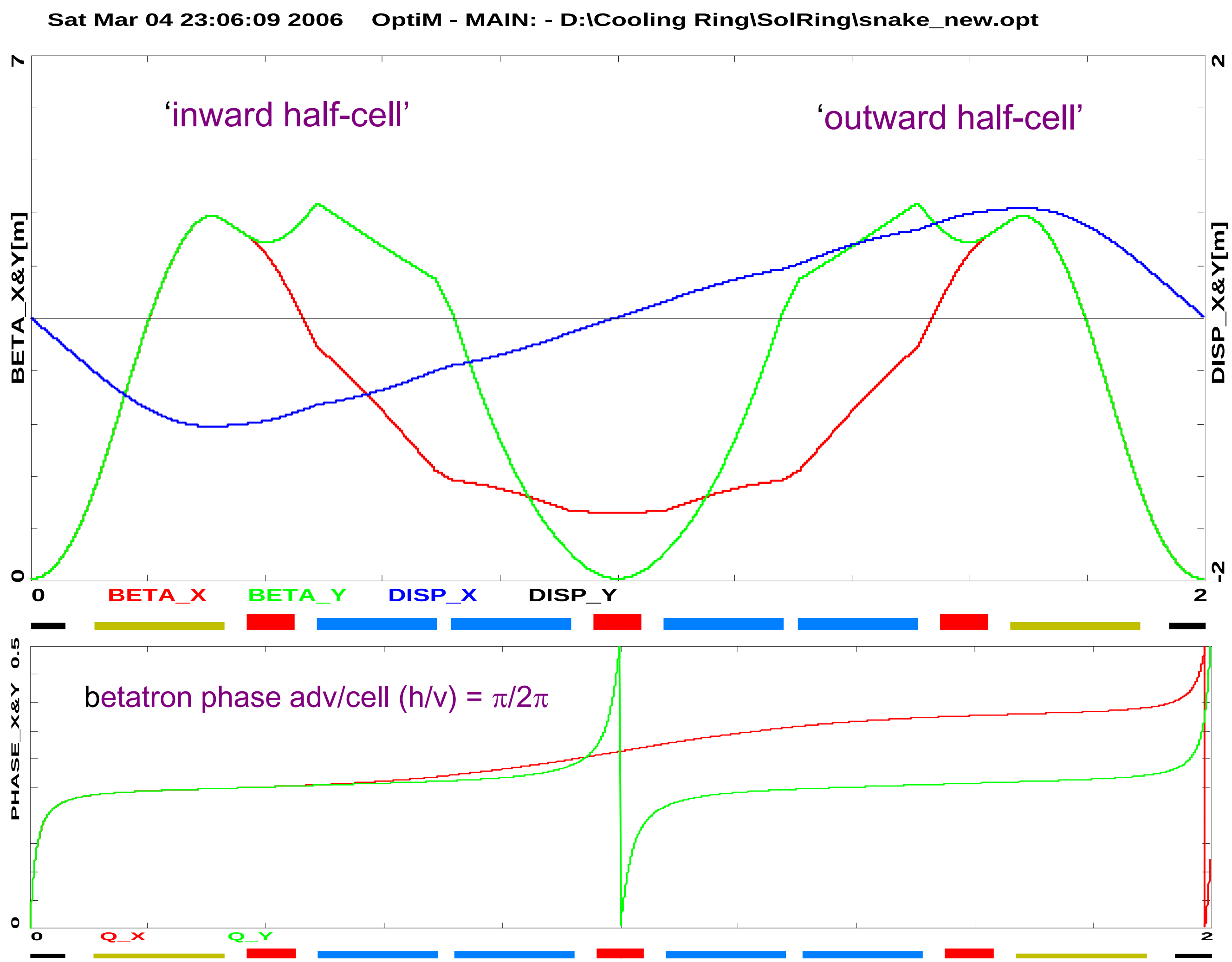
$$\alpha=-\frac{1}{2}\frac{d\beta}{ds}$$

Beta functions in axially symmetric FOFO cell



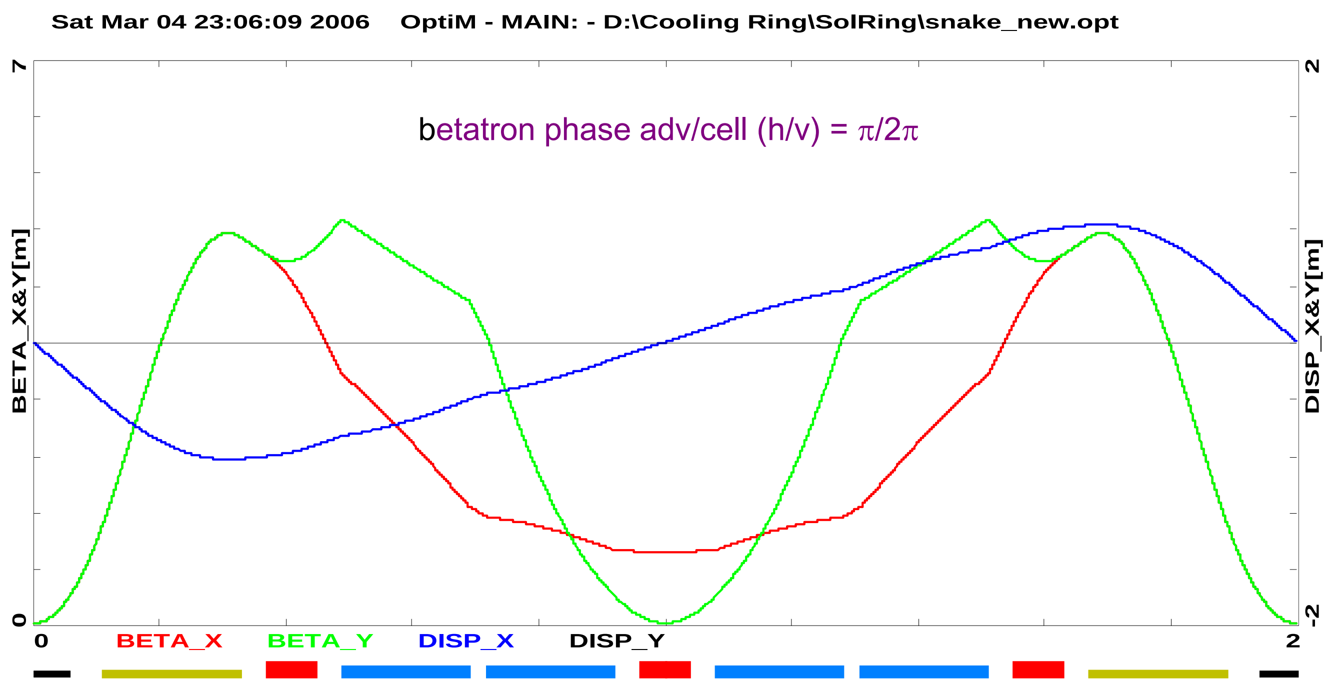
c1	L[cm]=130	B[kG]=38.1	Aperture[cm]=10
c2	L[cm]=80	B[kG]=-34.3	Aperture[cm]=10

Periodic Cell – Optics



Periodic Cell – Magnets

‘inward half-cell’ ‘outward half-cell’



solenoids:

L[cm]	B[kG]
22	105
22	105

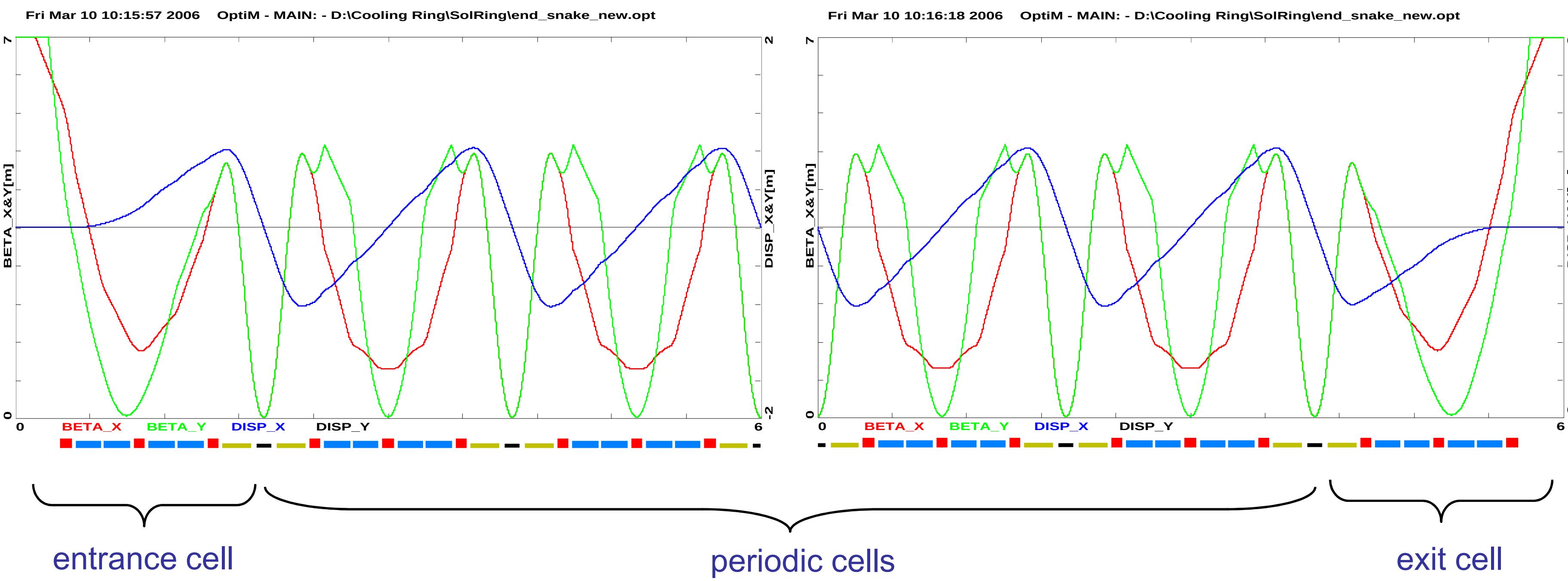
quadrupoles:

L[cm]	G[kG/cm]
8	1.79754
8	-0.3325
8	1.79754

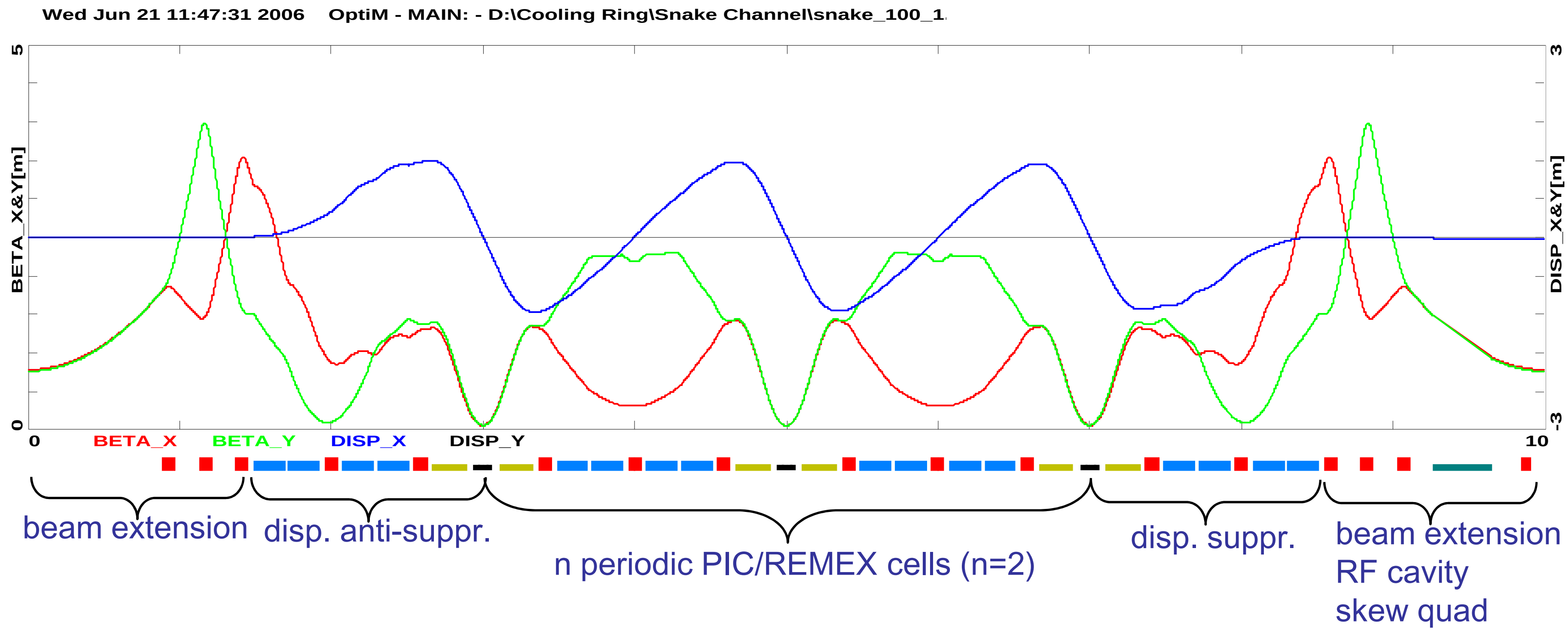
dipoles:

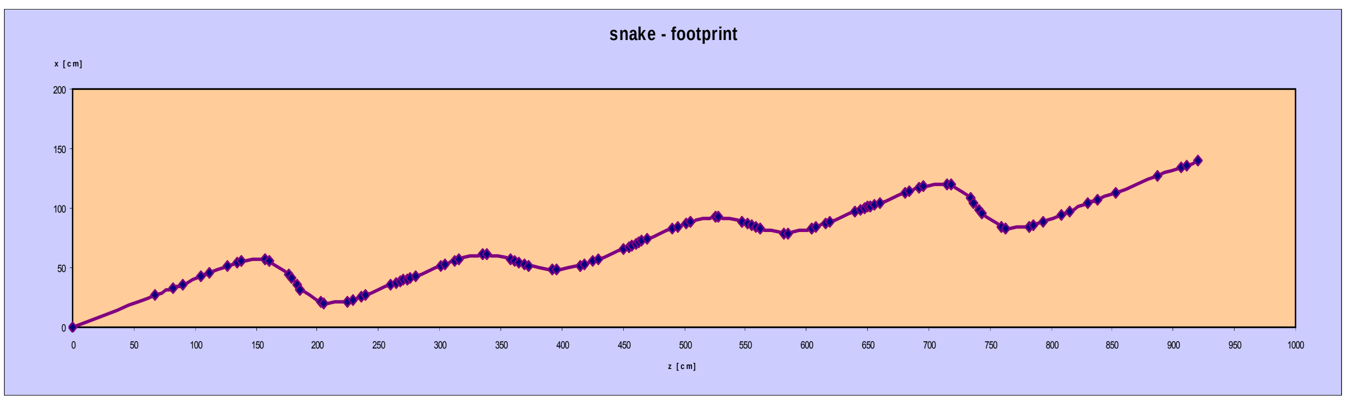
$L=20; => 20 cm
$B= 25; => 25 kGauss
$Ang=$L*$B/$Hr; => 0.4996 rad
$ang=$Ang*180/$PI; => 28.628 deg

‘Snake’ cooling channel – Dispersion suppression

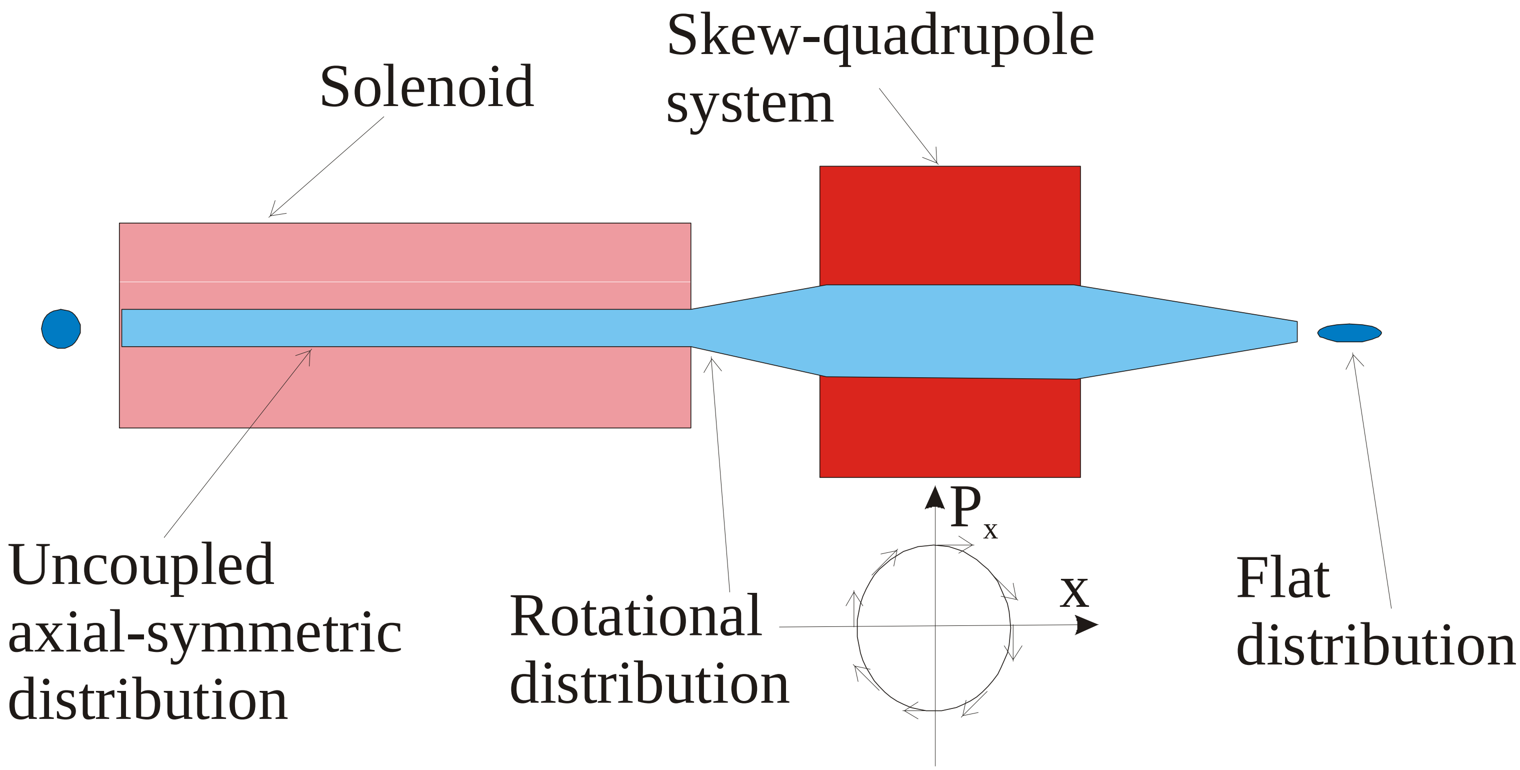


Muon Cooling Channel – Optics





Vertex-to-plane transformer insert



Vertex-to-plane transformer insert

- Eigen-vectors of the decoupled motion in the coordinate system rotated by 45^0

$$\begin{bmatrix} \sqrt{\beta_1} \\ -\dfrac{i+\alpha_1}{\sqrt{\beta_1}} \\ 0 \\ 0 \end{bmatrix} \xrightarrow[\textit{rotation}]{\text{45deg.}} \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{\beta_1} \\ -\dfrac{i+\alpha_1}{\sqrt{\beta_1}} \\ \sqrt{\beta_1} \\ -\dfrac{i+\alpha_1}{\sqrt{\beta_1}} \end{bmatrix} \xrightarrow[\alpha_1=2\alpha]{\beta_1=2\beta} \begin{bmatrix} \sqrt{\beta} \\ -\dfrac{i+2\alpha}{2\sqrt{\beta}} \\ \sqrt{\beta} \\ -\dfrac{i+2\alpha}{2\sqrt{\beta}} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{F}_2 \\ \mathbf{F}_2 \end{bmatrix}$$

  - Rotational eigen-vectors

$$\begin{bmatrix} \mathbf{F}_2 \\ i\mathbf{F}_2 \end{bmatrix} \qquad \begin{bmatrix} i\mathbf{F}_2 \\ \mathbf{F}_2 \end{bmatrix}$$

Vertex-to-plane transformer insert

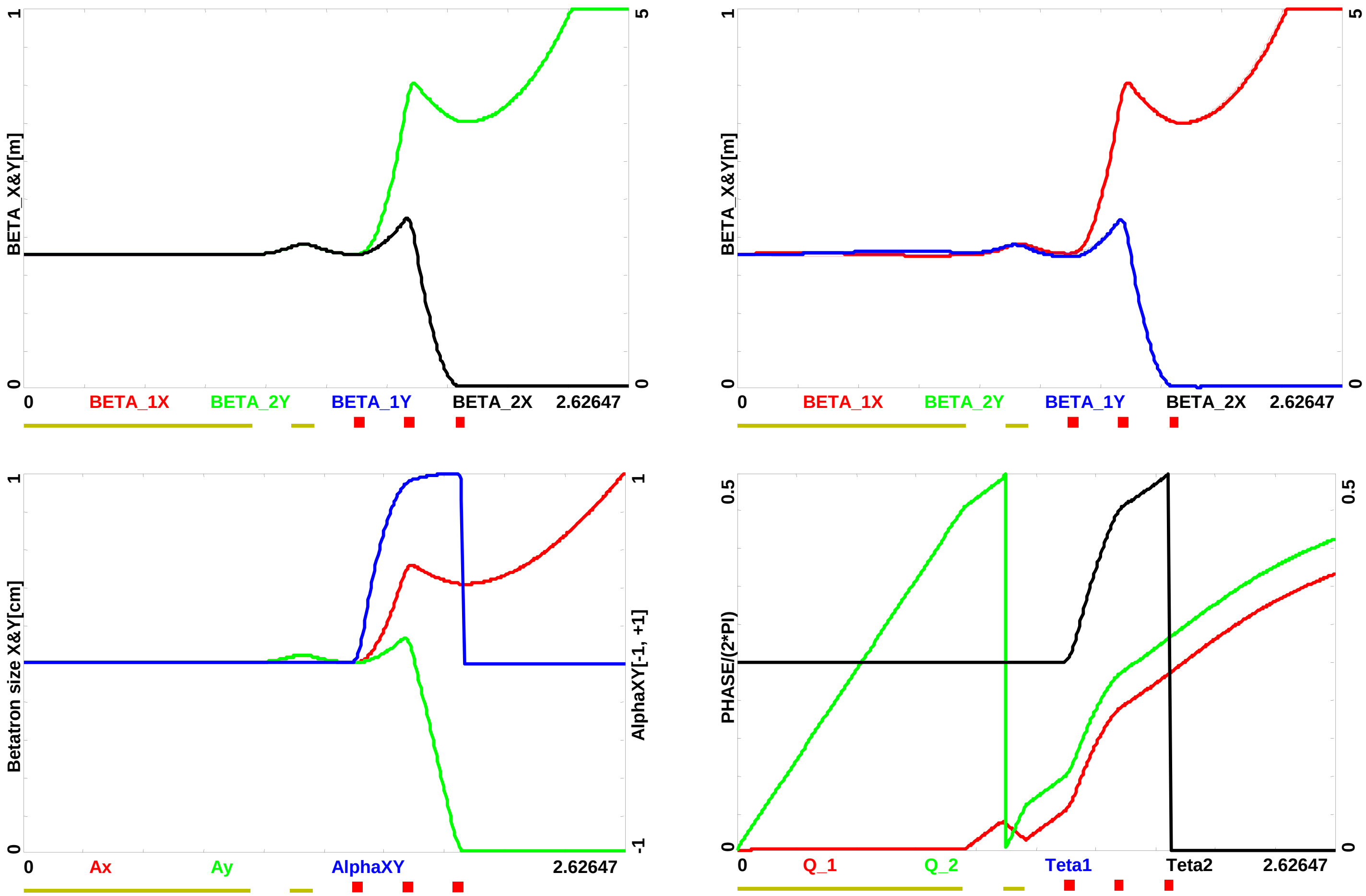
- Focusing system with 45^0 difference between the horizontal and vertical betatron phase advances will transform the initial vertex distribution into the flat one

- The resulting 2D emittances are as follows

$$\varepsilon_1 = \frac{\varepsilon_T}{\sqrt{1+\Phi^2\beta_0^{\ 2}} - \Phi\beta_0} \quad , \qquad \varepsilon_2 = \frac{\varepsilon_T}{\sqrt{1+\Phi^2\beta_0^{\ 2}} + \Phi\beta_0} \quad .$$

- Lattice implementation – Twiss functions, beam sizes etc.

Vertex-to-plane transformer insert



E_{kin} = 10 MeV, T_c = 0.2 eV, R_c = 0.5 cm, B_{sol} = 1 kG,

$\Rightarrow$ $\varepsilon_1 = 7.14 \cdot 10^{-3}$ cm, $\varepsilon_2 = 3.24 \cdot 10^{-8}$ cm

Summary

- Relationships between the eigen-vectors, beam emittances and the beam ellipsoid in 4D phase space
  - From the beam ellipsoid to the eigen-vectors (equivalence of both pictures)
- New parametrization of eigen-vectors in terms of generalized Twiss functions
  - Complete Weyl-like representation
    - 10 independent parameters to fully describe the motion
    - transport line ambiguities resolved
  - Developed software based on this representation allows effective analysis of coupled betatron motion for both circular accelerators and transfer lines (OptiM).