Nonlinear Beam Dynamics

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The Hamiltonian of Charged Particle in Electromagnetic Field

The Hamiltonian for a charged particle in an external electromagnetic field is given by

$$H(x, p_x, y, p_y, -p_t, ct; s) = -[1 + K(s)x] \times [\frac{e}{p_0} A_s(s) + \sqrt{1 - [p_x - \frac{e}{p_0} A_x(s)]^2 - [p_y - \frac{e}{p_0} A_y(s)]^2 - \frac{2p_t}{\beta} + p_t^2}] \quad (1)$$

$\beta = \frac{v}{c}$ Relativistic factor
$p_t = -\frac{\delta E}{p_0 c}$
$K(s) = \frac{1}{\rho(s)}$ ($\rho(s)$ radius of curvature)

Motion of Particles With Momentum Deviation

A particle with a momentum deviation $\delta \equiv \frac{p-p_o}{p_o}$ will have a Hamiltonian generated from the following canonical transformation

$$F_2 = \frac{ct}{\beta}[1 - \sqrt{1 + \beta^2 (2\delta + \delta^2)}] \tag{2}$$

$$H_2 = H_1 + \frac{\partial F_2}{\partial s} \tag{3}$$

$$-cT = \frac{\partial F_2}{\partial \delta} = -\frac{\beta (1+\delta) ct}{\sqrt{1 + \beta^2 (2\delta + \delta^2)}} \tag{4}$$

$$p_t = \frac{\partial F_2}{\partial (ct)} = \frac{1}{\beta}[1 - \sqrt{1 + \beta^2 (2\delta + \delta^2)}] \tag{5}$$

where $t = T \frac{\sqrt{1+\beta^2(2\delta+\delta^2)}}{\beta(1+\delta)}$

The new Hamiltonian

$$H(x, p_x, y, p_y, \delta, ct; s) = -[1 + K(s)x] \times [\frac{e}{p_0} A_s(s) + \sqrt{(1+\delta)^2 - [p_x - \frac{e}{p_0} A_x(s)]^2 - [p_y - \frac{e}{p_0} A_y(s)]^2}] \tag{6}$$

Where $1 - \frac{2}{\beta} p_t + p_t^2 = (1+\delta)^2$

Approximations

- Adiabatic approximation: momentum deviation can be viewed as a slowly varying parameter
- Considering the multipole expansion of the vector potential in the source-free region with piece-wise constant fields (dipoles with cylindrical geometry and quarupoles and sextupoles with Cartesian geometry)
- Considering the case where the local radius of curvature is small

$$A_x(s) = 0 \tag{7a}$$

$$A_y(s) = 0 \tag{7b}$$

$$\frac{e}{p_0} A_s(s) = -Re \sum_{n=1}^{\infty} \frac{1}{n} (ia_n(s) + b_n(s))(x + iy)^n \tag{7c}$$

Expansion of The Hamiltonian to third order in the phase space

$$H(x, p_x, y, p_y; s) = \frac{p_x^2 + p_y^2}{2(1+\delta)} - b_1(s)x\delta + \frac{b_1^2(s)}{2}x^2 + \frac{b_2(s)}{2}(x^2 - y^2) + \frac{b_3(s)}{3}(x^3 - 3xy^2) + O(4) \tag{8}$$

Equations of motion derived from Hamilton's equations

$$x' = \frac{\partial H}{\partial p_x} = \frac{p_x}{1+\delta} + O(3) \tag{9a}$$

$$p_x' = -\frac{\partial H}{\partial x} = b_1(s)\delta - (b_1^2(s) + b_2(s))x - b_3(s)(x^2 - y^2) + O(3) \tag{9b}$$

$$y' = \frac{\partial H}{\partial p_y} = \frac{p_y}{1+\delta} \tag{9c}$$

$$p_y' = -\frac{\partial H}{\partial y} = b_2(s)y + 2b_3(s)xy + O(3) \tag{9d}$$

$$\tag{9e}$$

Hill's Equations

Combining Hamilton's equations of motion into two second order ODE's, we could reach Hill's equations.

$$x'' + \frac{b_2(s) + b_1^2(s)}{1+\delta}x = b_1(s)\delta - \frac{b_3(s)}{1+\delta}(x^2 - y^2) + O(3) \quad (10a)$$

$$y'' - \frac{b_2(s)}{1+\delta}y = \frac{2b_3(s)}{1+\delta}xy + O(3) \quad (10b)$$

First Order Chromaticity

Origin of Chromaticity

- Deviations of particle energies from design energy cause perturbations in the solutions of the equations of motion
- Focusing errors due to energy error cause particles to be imaged at different focal points causing a blur of the beam spot

Tune chromaticity

The variation of tunes with energy $\xi = \frac{\Delta\nu}{\Delta p/p_0}$

First Order Chromaticity

Chromaticity in linear colliders

At the collision point of linear colliders the blur causes severe degradation of the attainable luminosity

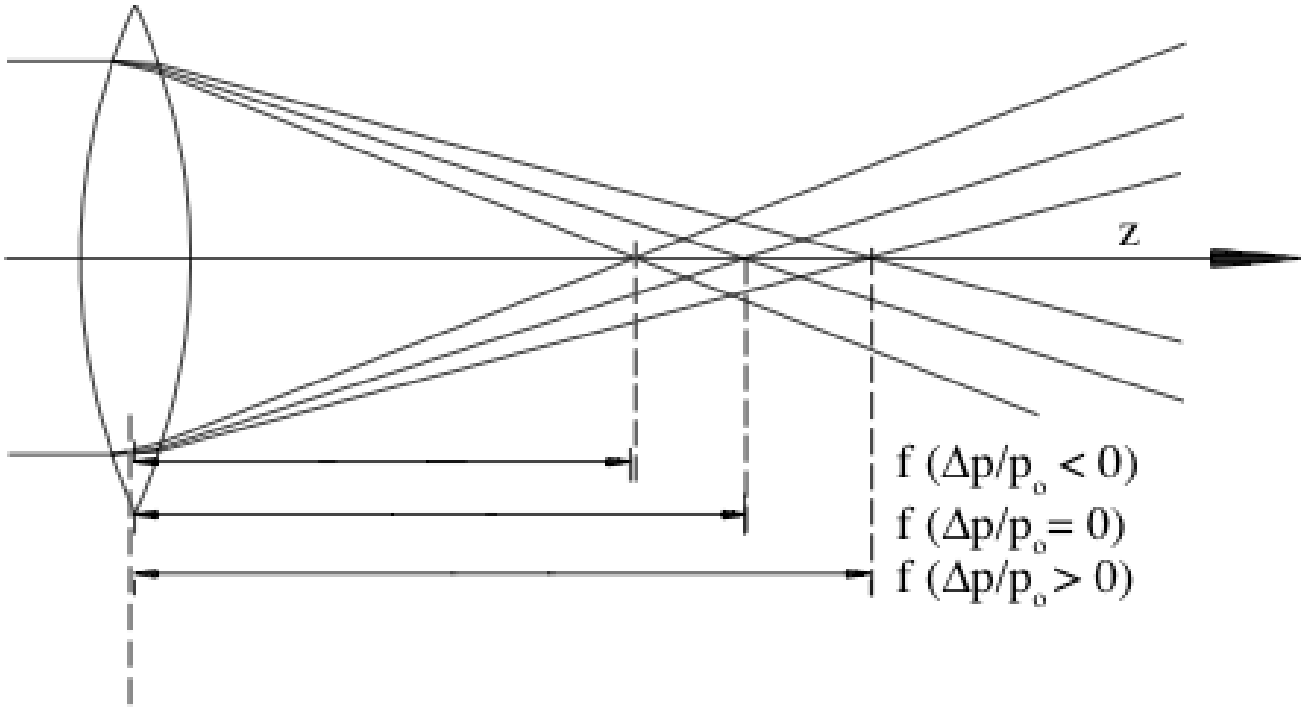
Chromaticity in circular colliders

The tune of the accelerator is determined by the overall focusing and tune errors occur when the focusing system is in error.

- Avoid loss of particles due to tune shifts into resonances
- Prevent beam loss due to headtail instability

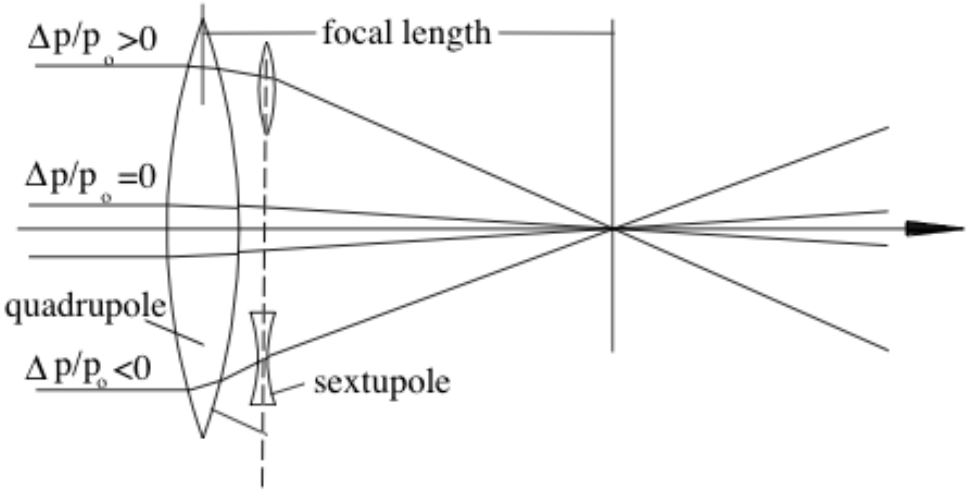
First Order Chromaticity

- The lowest order chromatic perturbation is caused by the variation of the focal length of the quadrupoles with energy



First Order Chromaticity

- Particles with different energies are separated by the introducing dispersion function.
- Once the particles are separated by energy we apply different focusing corrections depending on the energy of the particles
- Higher energy particles are focused less than ideal energy particles and lower energy particles are overfocused
- Sextupole magnet introduce focusing for higher energy particles and defocusing for lower energy particles



Derivation of First Order Chromaticity

From equation 10 considering the case for only quadrupole and sextupole magnets $b_1(s) = 0$
$\frac{1}{1+\delta} \approx (1 - \delta + ...)$

$$x'' + b_2(s)x(1-\delta) = -b_3(1-\delta)(x^2 - y^2) \tag{11a}$$
$$y'' - b_2(s)y(1-\delta) = 2b_3(1-\delta)xy \tag{11b}$$

NOTE

From now on we will change notations to be consistent with Weidemann's
Quadrupole field $k = b_2$
Sextupole field $m = 2b_3$

Derivation of First Order Chromaticity

To retain only betatron oscillation terms involving x_β or y_β to derive chromatic tune shifts.
Setting
$x = x_\beta + \eta_x \delta$
$y = y_\beta$ where $\eta_y = 0$
Three types of chromatic perturbation terms:

- Terms depending on the betatron motion only
- Terms depending on the momentum error only
- Terms depending on both

$$x'' + k(s)x(1-\delta) = -\frac{m}{2}(1-\delta)(x^2 - y^2) \tag{12a}$$

$$y'' - k(s)y(1-\delta) = m(1-\delta)xy \tag{12b}$$

$$x''_\beta + kx_\beta = kx_\beta\delta - m\eta_x x_\beta\delta - \frac{m}{2}(x_\beta^2 - y_\beta^2) + O(3) \tag{13a}$$

$$y''_\beta - ky_\beta = -ky_\beta\delta + m\eta_y y_\beta\delta + mx_\beta y_\beta) + O(3) \tag{13b}$$

Ignoring the nonchromatic terms of second order (geometric aberrations)

$$x''_\beta + kx_\beta = (k - m\eta_x)x_\beta\delta \tag{14a}$$

$$y''_\beta - ky_\beta = -(k - m\eta_y)y_\beta\delta \tag{14b}$$

The perturbation terms are linear in the betatron amplitude

$$\xi_x = -\frac{1}{4\pi}\oint \beta_x(k - m\eta_x)ds \tag{15a}$$

$$\xi_y = \frac{1}{4\pi}\oint \beta_y(k - m\eta_x)ds \tag{15b}$$

Natural Chromaticity

$$\xi_{x0} = -\frac{1}{4\pi} \oint \beta_x k ds \tag{16a}$$

$$\xi_{y0} = \frac{1}{4\pi} \oint \beta_y k ds \tag{16b}$$

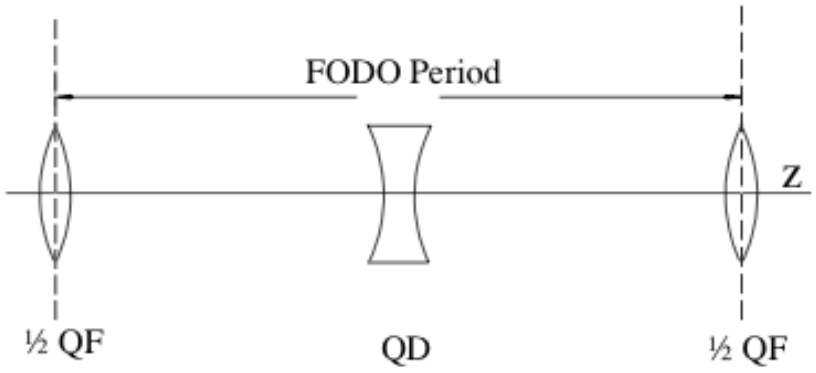
Natural chromaticities are always negative which is to be expected since focusing is less effective for higher energy particles ($\delta > 0$) and therefore the number of betatron oscillations is reduced.

Symmetric FDOD cell

FODO period ==> $1/2QF - L - QD - L - 1/2QF$
The transformation matrix with thin lens approximation

$$M_{FODO} = \begin{pmatrix} 1 - 2\frac{L^2}{f^2} & 2L(1 + \frac{L}{f}) \\ -\frac{1}{f^*} & 1 - 2\frac{L^2}{f^2} \end{pmatrix} \tag{17}$$

where $f = 1/kl$, $f_f = -f_d = f$, $1/f^* = 2(1 - L/f)L/f^2$



Symmetric FDOD cell

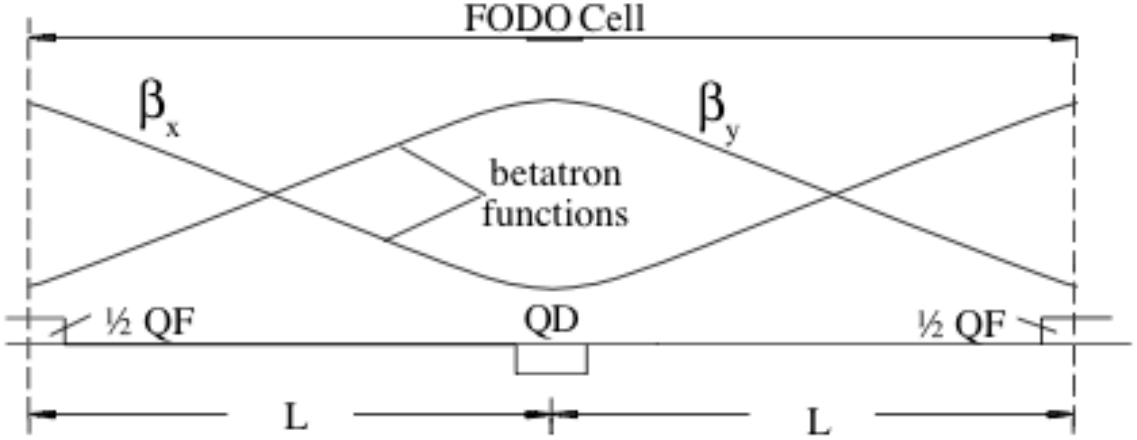
In the middle of QF $\beta_x = \beta^+, \beta_y = \beta^-$
In the middle of QD $\beta_x = \beta^-, \beta_y = \beta^+$

$$\beta^+ = L\frac{\kappa(\kappa+1)}{\sqrt{\kappa^2-1}} \tag{18a}$$

$$\beta^- = L\frac{\kappa(\kappa-1)}{\sqrt{\kappa^2-1}} \tag{18b}$$

$$sin\frac{\phi}{2} = \frac{1}{\kappa} \tag{18c}$$

$\kappa = \frac{f}{L}$ FODO parameter



Chromaticity of thin lens symmetric FODO Cell

Using what we learnt about FODO cells with thin lens approximation
The Chromaticity of one FODO half cell is

$$\xi_{x0} = -\frac{1}{4\pi}(\beta^+ \int k^+ ds + \beta^- \int k^- ds) = -\frac{(\beta^+ - \beta^-)}{4\pi} \int k ds \tag{19}$$

β^+ at center of focusing quadrupole
β^- at center of defocusing quadrupole
$k^+ = k^- = k$ quadrupole strength

Chromaticity of thin lens symmetric FODO Cell

From FODO cells slides we use

$$\beta^{+} = L\frac{\kappa(\kappa+1)}{\sqrt{\kappa^2-1}} \tag{20a}$$

$$\beta^{-} = L\frac{\kappa(\kappa-1)}{\sqrt{\kappa^2-1}} \tag{20b}$$

$\kappa = \frac{f}{L}$ FODO parameter and $\int kds = \frac{1}{f} = \frac{1}{(\kappa L)}$

$$\xi_{x0} = -\frac{1}{2\pi}\frac{1}{\sqrt{\kappa^2-1}} = -\frac{1}{\pi}tan(\psi_x/2) \tag{21}$$

The natural chromaticity of 90° FODO cell is $1/\pi$

Correcting Chromaticity

- Sextupoles must be placed along the orbit of a circular accelerator or along a beam transport line at locations where the dispersion function does not vanish
- the nonlinear nature of sextupole fields which causes dynamic instability for large amplitudes for which the sextupole field is no more a perturbation. The largest betatron oscillation amplitude which is still stable in the presence of nonlinear fields is called the dynamic aperture
- To maximize the dynamic aperture it is prudent to distribute many chromaticity correcting sextupoles along the beam line or circular accelerator
- To correct both the horizontal and the vertical chromaticity two different groups of sextupoles are required

Correcting Chromaticity

Using thin lens approximation, and replacing integrals in 15 by a sum the corrected chromaticities are

$$\xi_x = \xi_{x0} + \frac{1}{4\pi}(m_1\eta_{x1}\beta_{x1} + m_2\eta_{x2}\beta_{x2})l_s = 0 \tag{22a}$$

$$\xi_y = \xi_{y0} + \frac{1}{4\pi}(m_1\eta_{x1}\beta_{y1} + m_2\eta_{x2}\beta_{y2})l_s = 0 \tag{22b}$$

Assume that two different sextupoles (each of length l_s) at locations s_1 and s_2. Solving for the sextupole strengths

$$m_1 l_s = -\frac{4\pi}{\eta_{x1}}\frac{\xi_{x0}\beta y2 - \xi_{y0}\beta_{x2}}{\beta_{x1}\beta_{y2} - \beta_{x2}\beta_{y1}} \tag{23a}$$

$$m_2 l_s = -\frac{4\pi}{\eta_{x2}}\frac{\xi_{x0}\beta y1 - \xi_{y0}\beta_{x1}}{\beta_{x1}\beta_{y2} - \beta_{x2}\beta_{y1}} \tag{23b}$$

- Dispersion function at sextupoles should be large to minimize sextupoles strength
- Betatron functions must be $\beta_x >> \beta_y$ at m_1 sextupole and $\beta_x << \beta_y$ at the m_2 sextupole to avoid fighting between sextupoles leading to excessive strength requirements
- Only two sextupoles in a ring to correct chromaticities leads to very strong sextupoles causing both magnetic design problems and strong higher order aberrations
- Two families of sextupoles with individual magnets distributed evenly around the circular accelerator so the total required sextupole strength is spread over all sextupoles

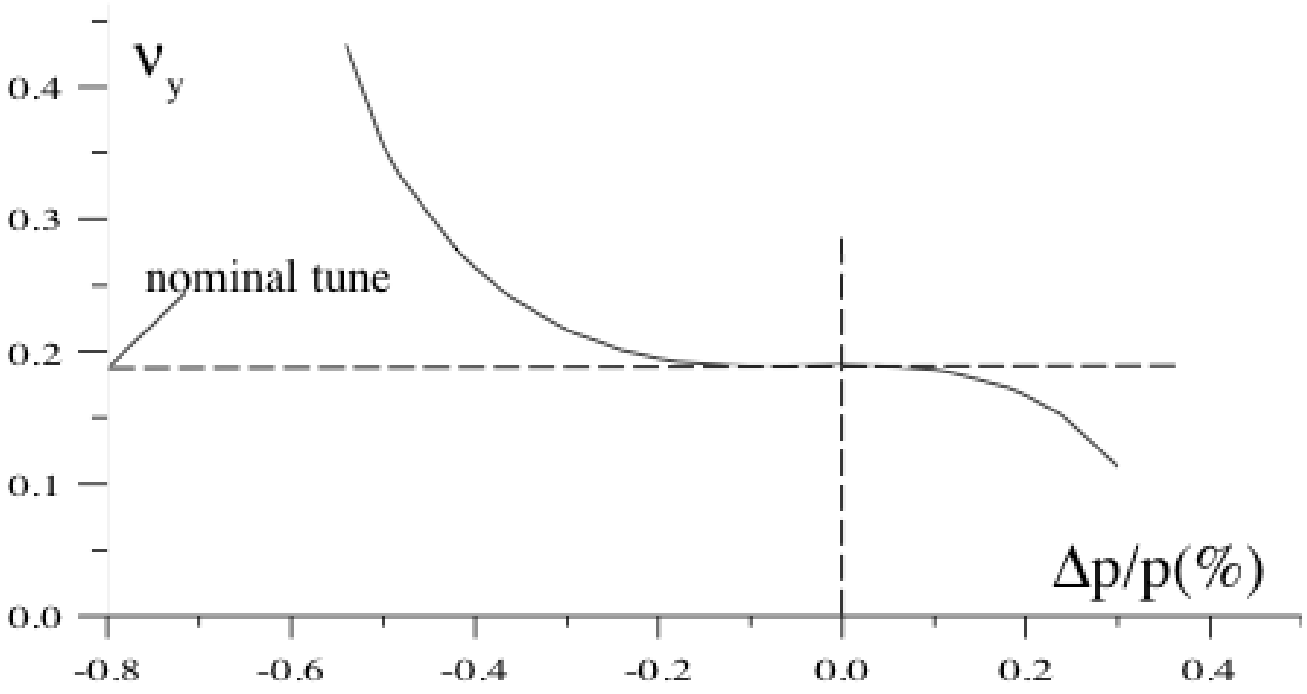
For the general case of chromaticity correction

$$\xi_x = \xi_{x0} + \frac{1}{4\pi} \sum_i m_i \eta_{xi} \beta_{xi} l_{si} \tag{24a}$$

$$\xi_y = \xi_{y0} + \frac{1}{4\pi} \sum_i m_i \eta_{xi} \beta_{yi} l_{si} \tag{24b}$$

The sum is taken over all sextupoles. In the case of a two family correction scheme we still can solve for m_1 and m_2 by grouping the terms into two sums

Variation of the vertical tune with energy in the storage ring PEP if the chromaticity is corrected by only two families of sextupoles



Sextupoles Separated by a -I Transformation

To overcome amplitude dependent aberration introduced by sextupoles
Sextupoles are identified in pairs
Each pair is separated by a negative unity transformation -I

$$-I = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{25}$$

Exercise

What is the phase advance in both planes given by this transformation?

The sextupoles are thin magnets introducing kicks on particle trajectories given by

$$\Delta x' = -\frac{1}{2} m_0 l_s (x^2 - y^2) \tag{26a}$$

$$\Delta y' = -m_0 l_s x y \tag{26b}$$

The (4×4) transformation matrix through a thin sextupole M_s

$$M_s(x_0, y_0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} m_0 l_s x_0 & 1 & \frac{1}{2} m_0 l_s x_0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & m_0 l_s x_0 & 1 \end{pmatrix} \tag{27}$$

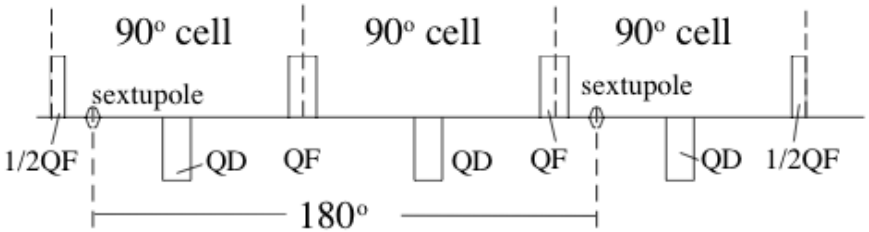
Evaluating the complete transformation through Sextupole pair separated by -I transformation

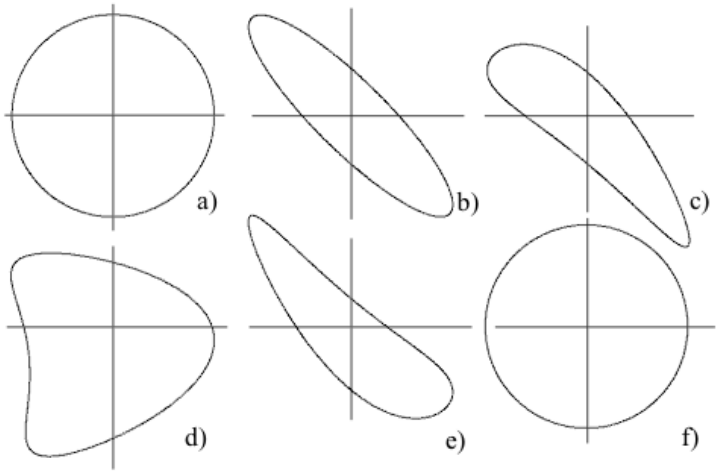
$$M_t = M_s(x_0, y_0)(-I)M_s(-x_0, -y_0) = -I \tag{28}$$

Shows a complete cancellation of geometric aberrations in both horizontal and vertical planes
This correction scheme has been applied successfully to the final focus system of the Stanford Linear Collider

Example of Chromaticity Correction

FODO lattice with 90° cells, $L_q = 3.6m$ integrated half quadrupole strength of $(kl_q)^{-1} = \sqrt{2}L_q$





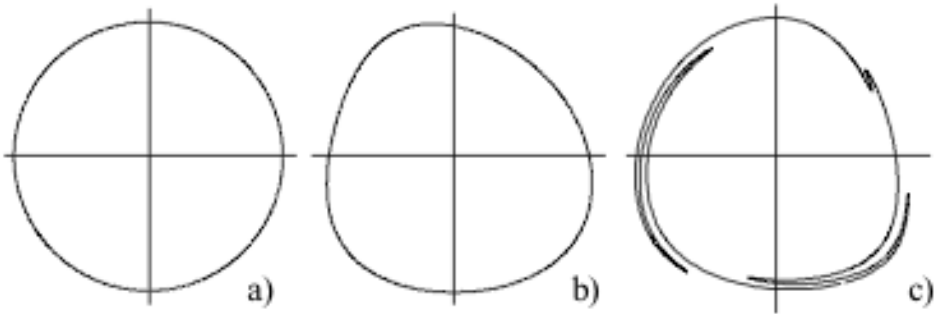
(a) Upright ellipse at the beginning of the FODO lattice
(b) Quad focusing tilted ellipse at the entrance to first sext
(c) The thin lens sextupole introduces significant angular perturbation
(d) Large lateral aberrations in the quad QF
(e) At the entrance to second sext the distorted phase ellipse is rotated by 180° and all aberrations are compensated again by this sextupole
(f) Ellipse at the end of the third FODO cell is upright with no distortions

Example of Chromaticity Correction

The evolution of the phase ellipse with a phase advance between the sextupole pair of 175^0

A distortion of the phase ellipse due to aberrations can be observed

Instability introduced as the particles pass through many similar cells



Montague Chromatic Function

Comparing the motion of particles with $\delta p/p$ to on momentum particles. Let

$$\beta_1 = \beta(0), \beta_2 = \beta(\delta) \tag{29a}$$
$$\alpha_1 = \alpha(0), \alpha_2 = \alpha(\delta) \tag{29b}$$
$$\phi_1 = \phi(0), \phi_2 = \phi(\delta) \tag{29c}$$

$$\Delta\beta = \beta_2 - \beta_1, \beta = \sqrt{\beta_1\beta_2} \tag{30a}$$
$$\Delta\phi = \phi_2 - \phi_1, \phi = \frac{1}{2}(\phi_1 + \phi_2) \tag{30b}$$
$$\Delta k = k_2(\delta) - k_1(0) \tag{30c}$$

k focusing gradient error

Montague Chromatic Function

Let us define the chromatic variables

$$A = \frac{\alpha_2\beta_1 - \alpha_1\beta_2}{\sqrt{\beta_1\beta_2}} \tag{31a}$$

$$B = \frac{\beta_2 - \beta_1}{\sqrt{\beta_2\beta_1}} = \frac{\Delta\beta}{\beta} \tag{31b}$$

Differntiation

$$\frac{dA}{ds} = \beta\Delta K + 2B\frac{d\phi}{ds} \tag{32a}$$

$$\frac{dB}{ds} = 2A\frac{d\phi}{ds} \tag{32b}$$

In a region with $\Delta k = 0$

$$\frac{d}{ds}(A^2 + B^2) = 0 \tag{33}$$

Therefore $(A^2 + B^2)$ is invariant
Montague Chromatic Function

$$W = \frac{1}{2}(A^2 + B^2)^{\frac{1}{2}}$$

Montague Chromatic Function

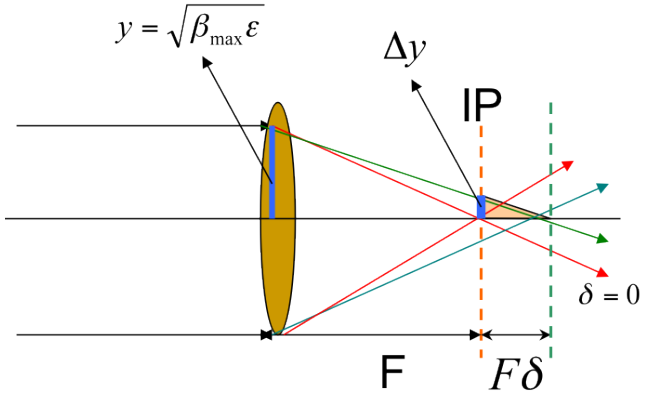
Montague Chromatic Function

$$W = \frac{1}{2}(A^2 + B^2)^{\frac{1}{2}} \tag{35}$$

- Dimensionless
- Absolute measure of the linear chromatic perturbation
- Factor of 1/2 due to chromatic function oscillate at twice betatron frequecny
- Useful in optimizing required sextupole strength for chromatic corrections (MAD-8 or MAD-X). especially local chromatic correction for low-β insertions as colliders final focus quads.

Spread of particle beam at interaction point

- In Collider ring accelerators the main source of chromaticity is the final focus quadrupoles (IR)
- This limits the achievable luminosity of any accelerator

Chromatic dilution of beam size

$$\frac{\Delta\sigma_y}{\sigma_y^*} = \frac{F}{\beta^*}\sigma_E \tag{36}$$