Accelerator Physics

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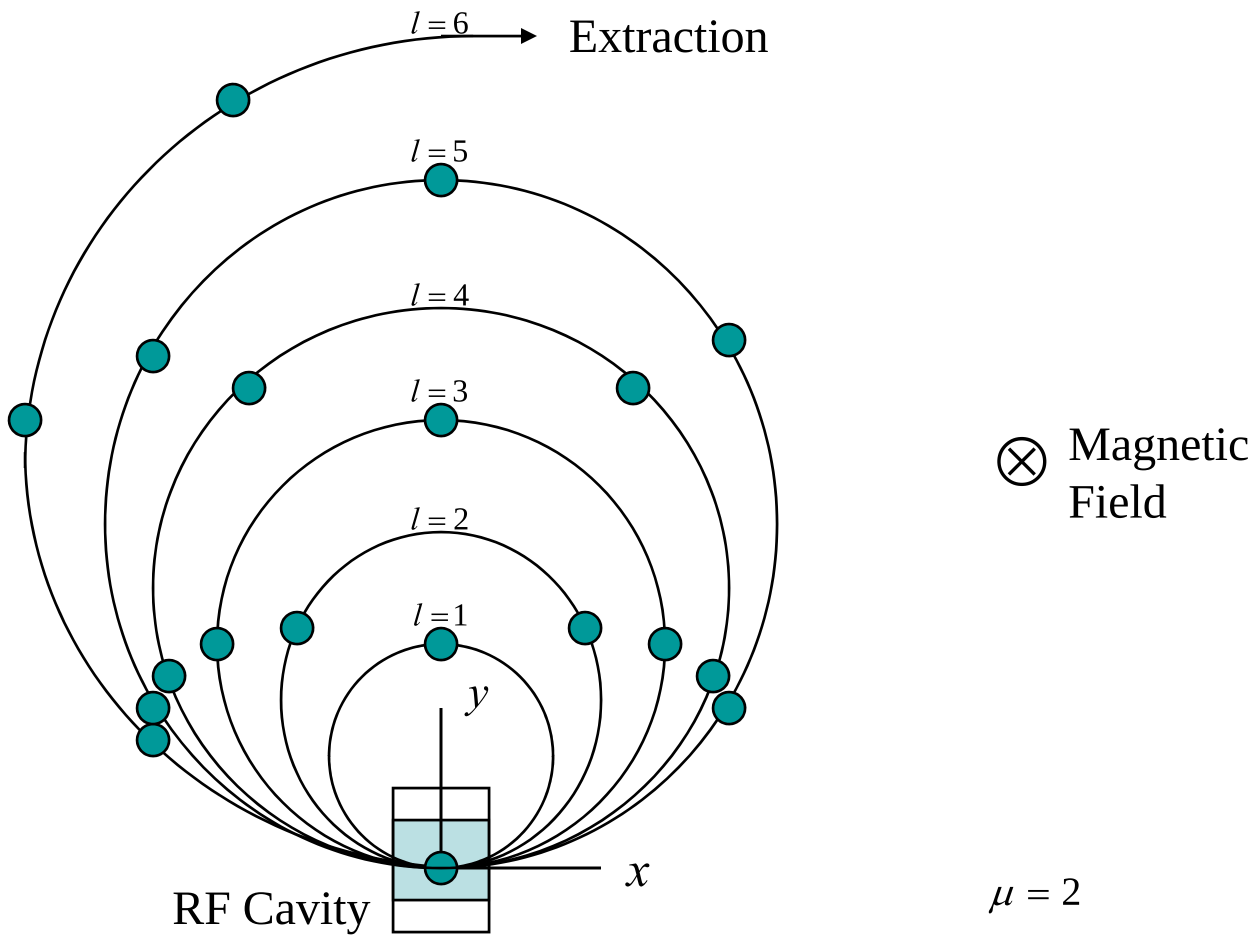
Lecture 4

Classical Microtron: Veksler (1945)

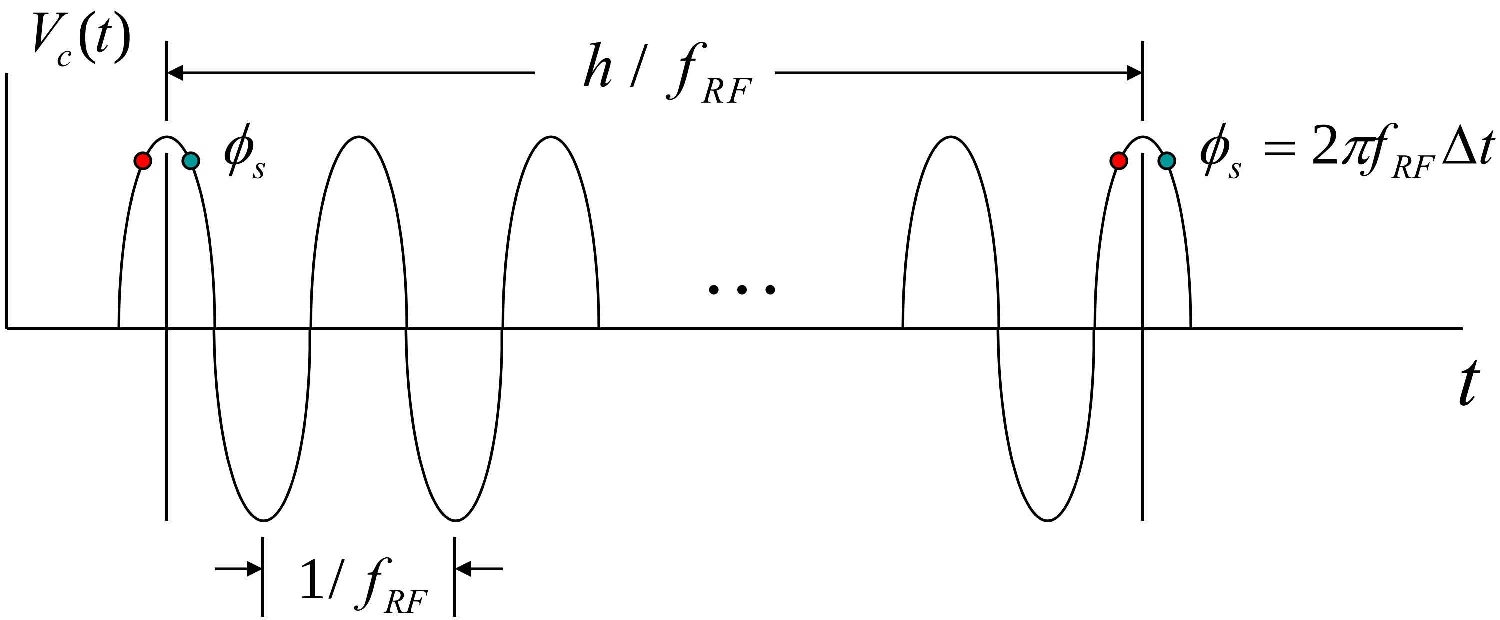


$\mu = 2$

$\nu = 1$

Synchrotron Phase Stability

Edwin McMillan discovered phase stability independently of Veksler and used the idea to design first large electron synchrotron.



$$h = L f_{RF} / \beta c$$

Harmonic number: # of RF oscillations in a revolution

Transition Energy

Beam energy where speed increment effect balances path length change effect on accelerator revolution frequency. Revolution frequency independent of beam energy to linear order. We will calculate in a few weeks

- Below Transistion Energy: Particles arriving EARLY get less acceleration and speed increment, and arrive later, with repect to the center of the bunch, on the next pass. Applies to heavy particle synchrotrons during first part of acceleration when the beam is non-relativistic and accelerations still produce velocity changes.

- Above Transistion Energy: Particles arriving EARLY get more energy, have a longer path, and arrive later on the next pass. Applies for electron synchrotrons and heavy particle synchrotrons when approach relativistic velocities. As seen before, Microtrons operate here.

Phase Stability Condition

"Synchronous" electron has

$$\text{Phase} = \phi_s \qquad\qquad E_l = E_o + leV_c \cos\phi_s$$

Difference equation for differences after passing through cavity pass $l + 1$:

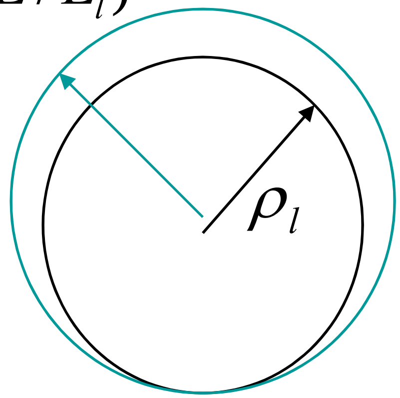
$$\begin{pmatrix} \Delta\phi_{l+1} \\ \Delta E_{l+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -eV_c \sin\phi_s & 1 \end{pmatrix} \begin{pmatrix} 1 & \dfrac{2\pi M_{56}}{\lambda E_l} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta\phi_l \\ \Delta E_l \end{pmatrix}$$

Because for an electron passing the cavity

$$\Delta E_{after} = \Delta E_{before} + eV_c\left(\cos(\phi_s + \Delta\phi) - \cos\phi_s\right)$$

Phase Stability Condition

$\rho_l(1+\Delta E/E_l)$



$$K_i = 1/\rho_i^2$$

$$D_{x,p,0} = \rho_i\left(1-\cos\left(s/\rho_i\right)\right) \qquad 0 \le s \le 2\pi\rho_i$$

$$\therefore\ M_{56} = \int \frac{D}{\rho} ds = \int_0^{2\pi\rho_l} \left(1-\cos s/\rho_l\right) ds$$

$$= 2\pi\rho_l$$

$$\begin{pmatrix} \Delta\phi_{l+1} \\ \Delta E_{l+1} \end{pmatrix} \approx \begin{pmatrix} 1 & \dfrac{4\pi^2\rho_l}{\lambda E_l} \\ -eV_c \sin\phi_s & 1-\dfrac{4\pi^2\rho_l eV_c}{\lambda E_l}\sin\phi_s \end{pmatrix} \begin{pmatrix} \Delta\phi_l \\ \Delta E_l \end{pmatrix}$$

Phase Stability Condition

Have Phase Stability if

$$-1 < \left(\frac{\mathrm{Tr}\, M}{2} \right) < 1 \rightarrow -1 < 1 - \frac{2\pi^2 \rho_l e V_c}{\lambda E_l} \sin\phi_s < 1$$

$$\frac{2\pi^2 \rho_l e V_c}{\lambda E_l} \sin\phi_s = \frac{\pi f_{RF} e V_c}{f_c m c^2} \cos\phi_s \tan\phi_s = \frac{\pi f_{RF} \Delta\gamma}{f_c} \tan\phi_s$$

i.e.,

$$\boxed{0 < \nu\pi \tan\phi_s < 2}$$

Phase Stability Condition

Have Phase Stability if

$$\left(\frac{\mathrm{Tr}\, M}{2}\right)^2 < 1$$

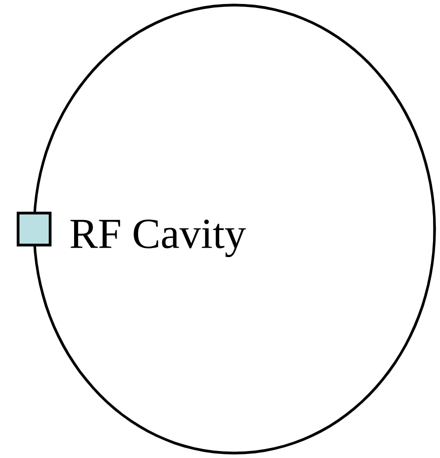
i.e.,

$$\boxed{0 < \nu\pi \tan\phi_s < 2}$$

Synchrotrons

Two basic generalizations needed

- Acceleration of non-relativistic particles
- Difference equation describing per turn dynamics becomes a differential equation with solution involving a new frequency, the synchrotron frequency



Acceleration of non-relativistic particles

For microtron, racetrack microtron and other polytrons, electron speed is at the speed of light. For non-relativistic particles the recirculation time also depends on the longitudinal velocity $v_z = \beta_z c$.

$$t_{recirc} = L / \beta_z c$$

$$\Delta t = \frac{\partial L}{\partial p}\frac{\Delta p}{\beta_z c} + \frac{L}{c}\frac{\partial}{\partial p}\left[\frac{1}{\beta_z}\right]\Delta p$$

$$\frac{\Delta t}{t_{recirc}} = \frac{M_{56}}{L}\frac{\Delta p}{p} - \frac{\Delta\beta_z}{\beta_z} = \frac{M_{56}}{L}\frac{\Delta p}{p} - \frac{1}{\gamma^2}\frac{\Delta p}{p}$$

Momentum Compaction $\alpha = (\Delta L / L)/(\Delta p / p) = M_{56} / L$

$$\frac{\Delta t}{t_{recirc}} = -\eta_c \frac{\Delta p}{p} \rightarrow \eta_c = \frac{1}{\gamma^2} - \frac{M_{56}}{L} = \frac{1}{\gamma^2} - \alpha$$

$$2p\Delta p c^2 = 2E\Delta E \rightarrow \frac{\Delta p}{p} = \frac{1}{\beta_z^2}\frac{\Delta E}{E} \rightarrow \frac{\Delta t}{t_{recirc}} = -\frac{\eta_c}{\beta_z^2}\frac{\Delta E}{E}$$

Transition Energy: Energy at which the change in the once around time becomes independent of momentum (energy)

$$\eta_c = 0 \rightarrow \frac{1}{\gamma_t^2} = \frac{M_{56}}{L} = \alpha$$

No Phase Focusing at this energy!

Equation for Synchrotron Oscillations

$$\begin{pmatrix} \Delta\phi_{l+1} \\ \Delta E_{l+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -eV_c \sin\phi_s & 1 \end{pmatrix} \begin{pmatrix} 1 & -\dfrac{2\pi L\eta_c}{\lambda\beta_z^2 E_l} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta\phi_l \\ \Delta E_l \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\dfrac{2\pi L\eta_c}{\lambda\beta_z^2 E_l} \\ -eV_c \sin\phi_s & 1 + \dfrac{2\pi L\eta_c}{\lambda\beta_z^2 E_l} eV_c \sin\phi_s \end{pmatrix} \begin{pmatrix} \Delta\phi_l \\ \Delta E_l \end{pmatrix}$$

Assume momentum slowly changing (adiabatic acceleration)
Phase advance per turn is

$$\cos\Delta\mu = 1 + \frac{\pi L\eta_c}{\lambda\beta_z^2 E_l} eV_c \sin\phi_s \rightarrow \Delta\mu^2 \approx -\frac{2\pi L\eta_c}{\lambda\beta_z^2 E_l} eV_c \sin\phi_s$$

So change in phase per unit time is

$$\frac{\Delta\mu}{T_0} \approx \frac{1}{T_0}\sqrt{-\frac{2\pi L\eta_c}{\lambda\beta_z pc} eV_c \sin\phi_s}$$

yielding synchrotron oscillations with frequency

$$\omega_s = \omega_{rev}\sqrt{-\frac{h\eta_c}{2\pi}\frac{eV_c}{pc}\sin\phi_s}$$

where the *harmonic number* $h = L/\beta_z\lambda$, gives the integer number of *RF* oscillations in one turn

Phase Stable Acceleration

At energies below transition, $\eta_c > 0$. To achieve acceleration with phase stability need $\phi_s < 0$

$$\therefore \omega_s = \omega_{rev}\sqrt{\frac{h\eta_c}{2\pi}\frac{eV_c}{pc}\sin\left(-\phi_s\right)}$$

At energies above transition, $\eta_c < 0$, which corresponds to the case we're used to from electrons. To achieve acceleration with phase stability need $\phi_s > 0$

$$\therefore \omega_s = \omega_{rev}\sqrt{\frac{h\left(-\eta_c\right)}{2\pi}\frac{eV_c}{pc}\sin\phi_s}$$

Large Amplitude Effects

Can no longer linearize the energy error equation.

$$\Delta\phi_{l+1} = \Delta\phi_l - \frac{2\pi L\eta_c}{\lambda\beta_z^2 E_l}\Delta E_l$$

$$\Delta E_{l+1} = \Delta E_l + eV_c\left(\cos\left(\phi_s + \Delta\phi_l\right) - \cos\phi_s\right)$$

$$\frac{d\Delta\phi}{dt} \approx \frac{\Delta\phi_{l+1} - \Delta\phi_l}{T_0} = -\frac{2\pi\eta_c}{\lambda p}\Delta E$$

$$\frac{d\Delta E}{dt} \approx \frac{\Delta E_{l+1} - \Delta E_l}{T_0} = \frac{eV_c\left(\cos\left(\phi_s + \Delta\phi_l\right) - \cos\phi_s\right)}{T_0}$$

$$\frac{d^2\Delta\phi}{dt^2} = -\frac{2\pi\eta_c}{\lambda p T_0} eV_c\left(\cos\left(\phi_s + \Delta\phi\right) - \cos\phi_s\right)$$

Constant of Motion (Longitudinal "Hamiltonian")

$$\frac{d\Delta\phi}{dt}\frac{d^2\Delta\phi}{dt^2} = -\frac{2\pi\eta_c}{\lambda p T_0} eV_c \frac{d\Delta\phi}{dt}\left(\cos\left(\phi_s + \Delta\phi\right) - \cos\phi_s\right)$$

$$\frac{1}{2}\left(\frac{d\Delta\phi}{dt}\right)^2 = -\frac{2\pi\eta_c}{\lambda p T_0} eV_c \left(\sin\left(\phi_s + \Delta\phi\right) - \Delta\phi\cos\phi_s\right) + C$$

$$H\left(\Delta\phi, T_0\Delta E\right) = \frac{1}{2}\frac{2\pi\eta_c}{\lambda p T_0}\left(T_0\Delta E\right)^2 + eV_c\left(\sin\left(\phi_s + \Delta\phi\right) - \Delta\phi\cos\phi_s\right)$$

Equations of Motion

If neglect the slow (adiabatic) variation of p and T_0 with time, the equations of motion approximately Hamiltonian

$$\frac{d\Delta\phi}{dt} = \frac{\partial H}{\partial\left(T_0 \Delta E\right)} \qquad \frac{d\left(T_0 \Delta E\right)}{dt} = -\frac{\partial H}{\partial \Delta\phi}$$

In particular, the Hamiltonian is a constant of the motion

Kinetic Energy Term

$$T = \frac{1}{2}\frac{2\pi\eta_c T_0}{\lambda p}\left(\Delta E\right)^2$$

Potential Energy Term

$$V = eV_c\left(\sin\left(\phi_s + \Delta\phi\right) - \Delta\phi\cos\phi_s\right)$$

No Acceleration

$$\phi_s = \pm\pi/2 \qquad\qquad V = eV_c \cos\Delta\phi$$

$$\frac{d^2\Delta\phi}{dt^2} = \omega_s^2 \sin\Delta\phi$$

Better known as the real pendulum.

With Acceleration

$$\frac{d^2\Delta\phi}{dt^2} = \frac{\omega_s^2}{\sin\phi_s}\left(\cos\left(\phi_s + \Delta\phi\right) - \cos\phi_s\right)$$

$$\frac{1}{2}\left(\frac{d\Delta\phi}{dt}\right)^2 = \frac{\omega_s^2}{\sin\phi_s}\left(\sin\left(\phi_s + \Delta\phi\right) - \Delta\phi\cos\phi_s\right) + C$$

Equation for separatrix yields “fish” diagrams in phase space. Fixed points at

$$\cos\left(\phi_s + \Delta\phi\right) = \cos\phi_s \qquad \Delta\phi = 0, -2\phi_s$$