Radiation Damping - Low emittance lattices

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Outline

- Beam dynamics with synchrotron radiation:
  - Discuss the effect of synchrotron radiation on the (linear) motion of particles in storage rings.
  - Define action-angle variables for describing symplectic motion of a particle along a beam line.
  - Derive expressions for the damping times of the vertical, horizontal and longitudinal emittances.
  - Introduce the synchrotron radiation integrals (Sand's Integrals).
  - Discuss the effects of quantum excitation, and derive expressions for the equilibrium horizontal and longitudinal beam emittances in an electron storage ring.

- M. Sands, "The physics of electron storage rings, an introduction" SLAC-121. 1970
- A. Wolski, University of Liverpool and the Cockcroft Institute, CAS 2009, **http://cas.web.cern.ch/cas/Germany2009/Lectures/PDF-Web/Wolski-1.pdf**

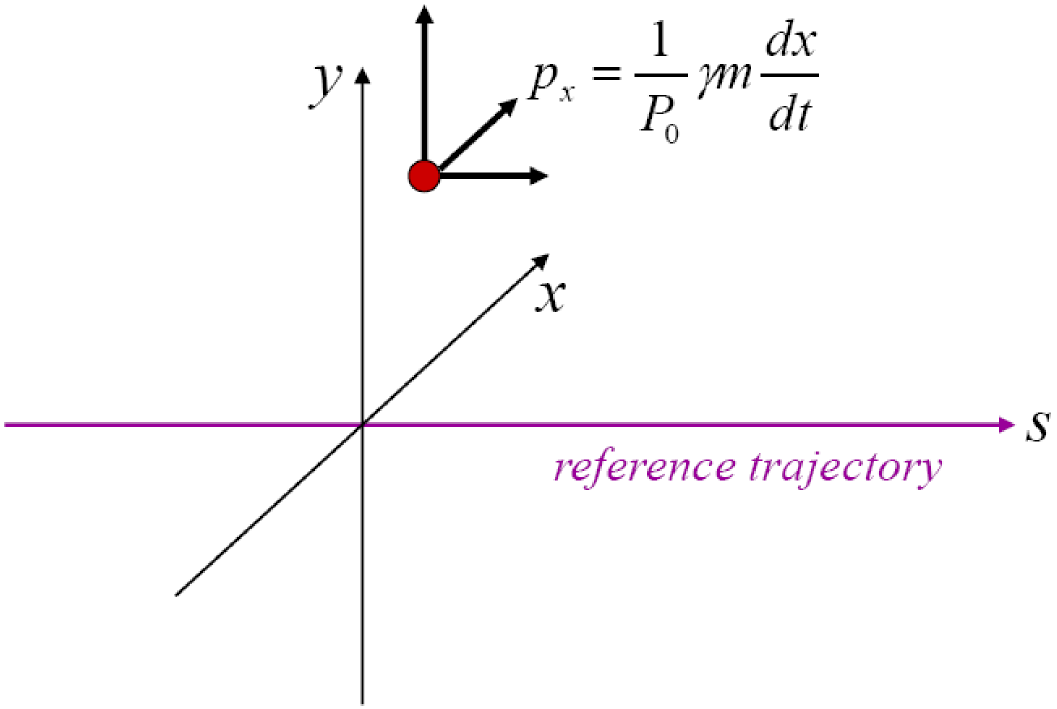
Outline – cont.

- Equilibrium emittance and storage ring lattice design
- Emittance preserving lattices :
  - The natural emittance for different types of lattice - Examples:
    - FODO
    - Double Bend Achromat (DBA)
    - Theoretical Minimum Emittance (TME)

Coordinate system

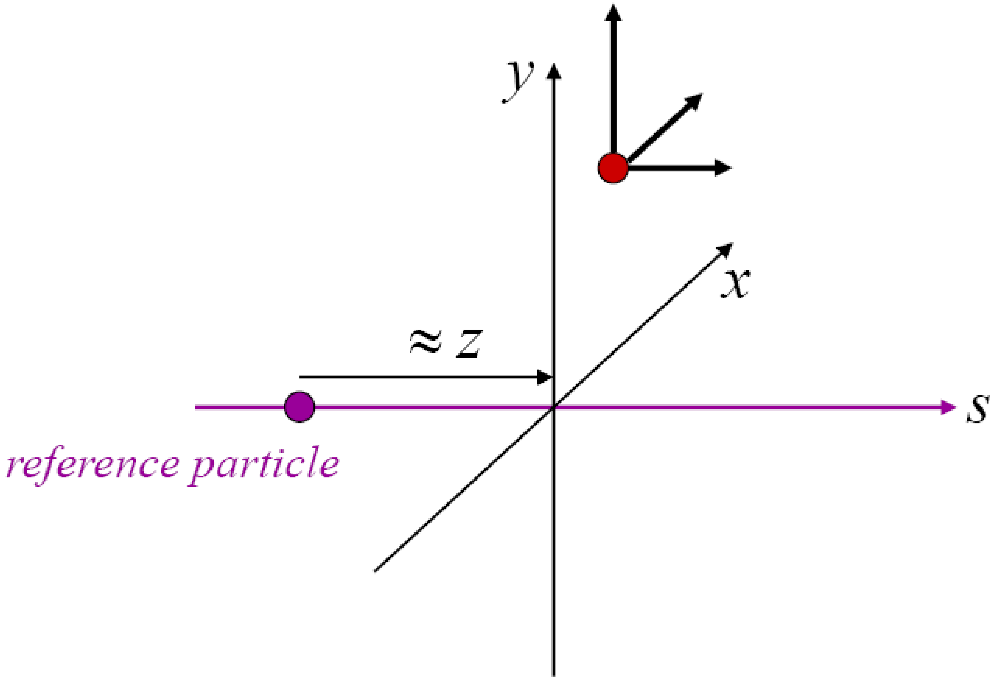
$$p_y = \frac{1}{P_0}\gamma m \frac{dy}{dt}$$

$$p_x = \frac{1}{P_0}\gamma m \frac{dx}{dt}$$



P_0 = reference momentum

Longitudinal coordinate

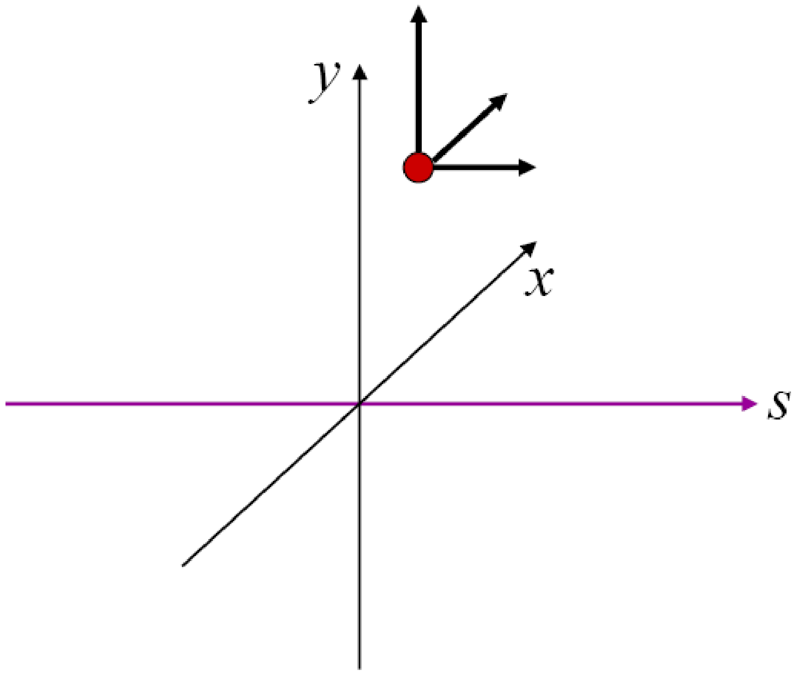


The reference particle is a particle travelling along the reference trajectory with momentum P_0 and velocity $\beta_0 c$.

If a particle is time τ *ahead of* the reference particle, then the longitudinal coordinate z is defined by:

$$z = c\tau$$

Energy deviation



If the particle has total energy E, then the energy deviation δ is defined by:

$$\delta = \frac{E}{P_0 c} - \frac{1}{\beta_0}$$

For ultra-relativistic particles ($\beta \approx \beta_0 \approx 1$), we have: $\delta \approx \frac{\Delta E}{E_0}$

Canonical variables

With the definitions in the previous slides, the coordinates and momenta form *canonical conjugate pairs*:

$$(x, p_x) \qquad (y, p_y) \qquad (z, \delta)$$

What this means, is that if M represents the linear transfer matrix for a beam line consisting of some sequence of drifts, solenoids, dipoles, quadrupoles, or RF cavities, i.e.:

$$\begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix}_{s=s_1} = M(s_1; s_0) \cdot \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ \delta \end{pmatrix}_{s=s_0}$$

then, neglecting radiation from the particle, the matrix M is *symplectic*.

Symplectic matrices

Mathematically, a matrix M is symplectic if it satisfies the relation:

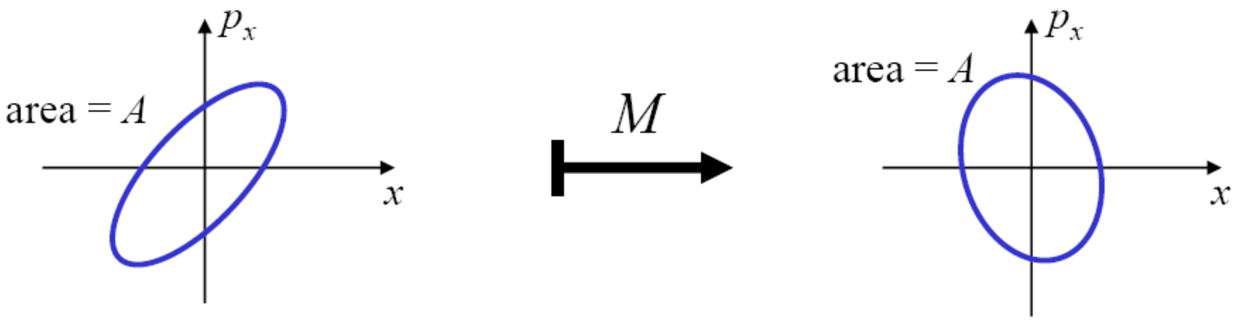
$$M^{\mathrm{T}} \, \mathrm{U} \, M = \mathrm{U}$$

where U is the antisymmetric matrix:

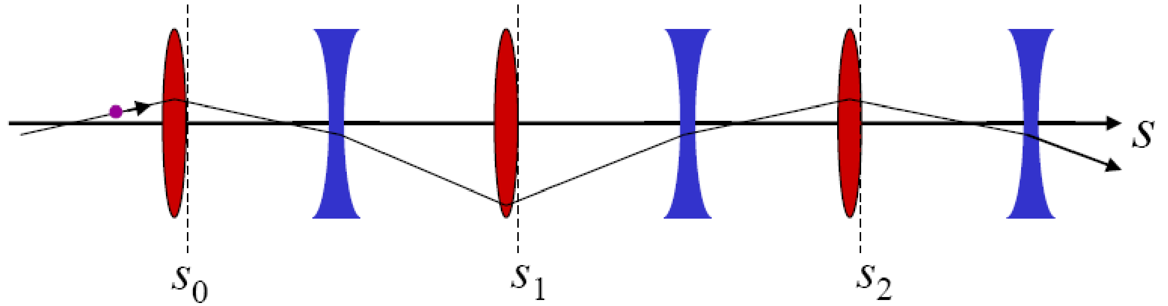
$$\mathrm{U} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

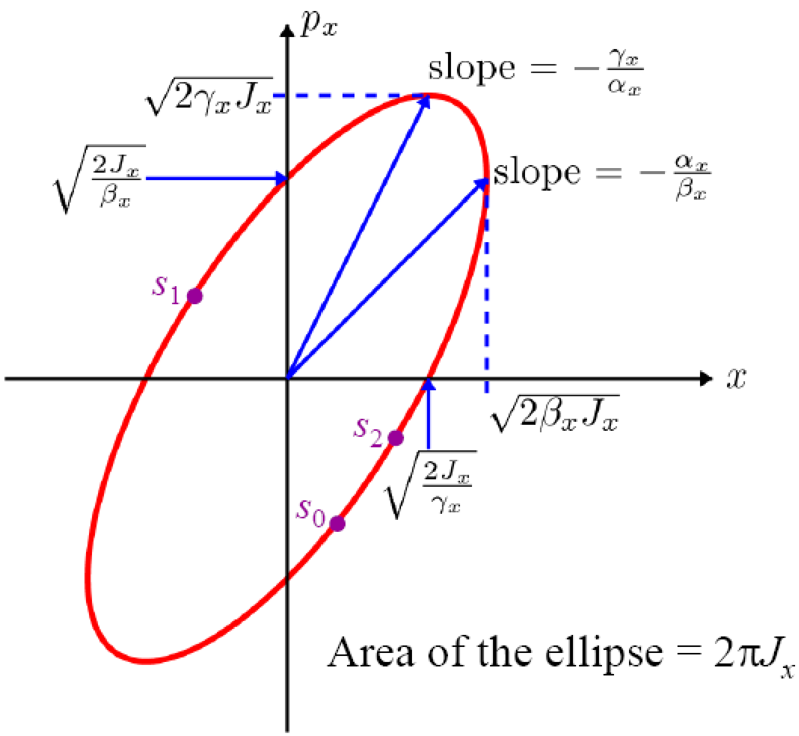
Physically, symplectic matrices preserve areas in phase space.

For example, in one degree of freedom:



Twiss parameters and the particle action





In an *uncoupled* periodic beam line, particles trace out ellipses in phase space with each pass through the periodic cell. The shape of the ellipse defines the *Twiss parameters* at the observation point.

The area of the ellipse defines the *action* J_x of the particle.

The action is the amplitude of the motion of the particle as it moves along the beam line.

Cartesian variables and action-angle variables

Applying simple geometry to the phase space ellipse, we find that the action (for uncoupled motion) is related to the Cartesian variables for the particle by:

$$2J_x = \gamma_x x^2 + 2\alpha_x x p_x + \beta_x p_x^2$$

We also define the angle φ_x as follows:

$$\tan\varphi_x = -\beta_x \frac{p_x}{x} - \alpha_x$$

The action-angle variables provide an alternative to Cartesian variables for describing the dynamics of a particle moving along a beam line. The advantage of action-angle variables is that, under symplectic transport, the action of a particle is constant.

It turns out that the action-angle variables are canonically conjugate.

Note: if the beam line is coupled, then we need to make a coordinate transformation to the "normal mode" coordinates, in which the motion in one mode is independent of the motion in the other modes. Then we can apply the equations as above.

Action and Emittance

The *action* J_x is a variable used to describe the amplitude of the motion of an individual particle. In terms of the action-angle variables, the Cartesian coordinate and momentum can be written:

$$x = \sqrt{2\beta_x J_x} \cos\varphi_x \qquad p_x = -\sqrt{\frac{2J_x}{\beta_x}} (\sin\varphi_x + \alpha_x \cos\varphi_x)$$

The *emittance* ε_x is the average amplitude of all particles in a bunch:

$$\varepsilon_x = \langle J_x \rangle$$

With this relationship between the emittance and the average action, we can obtain the following familiar relationships for the second-order moments of the bunch:

$$\langle x^2 \rangle = \beta_x \varepsilon_x \qquad \langle x p_x \rangle = -\alpha_x \varepsilon_x \qquad \langle p_x^2 \rangle = \gamma_x \varepsilon_x$$

Again, this is true for *uncoupled* motion.

Action and Radiation

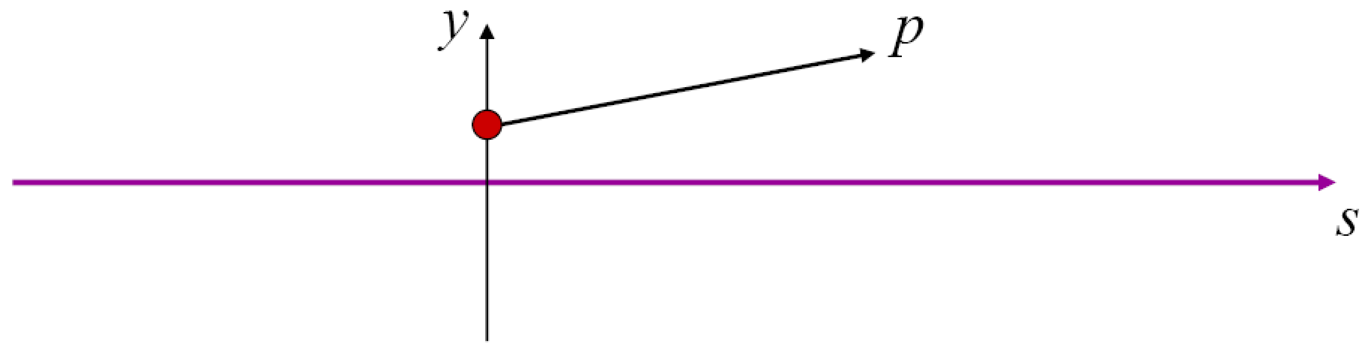
So far, we have considered only symplectic transport, i.e. motion of a particle in the electromagnetic fields of drifts, dipoles, quadrupoles etc. without any radiation.

However, we know that a charged particle moving through an electromagnetic field will (in general) undergo acceleration, and a charged particle undergoing acceleration will radiate electromagnetic waves.

What impact will the radiation have on the motion of the particle?

In answering this question, we will consider first the case of uncoupled vertical motion – for a particle in a storage ring, this turns out to be the simplest case.

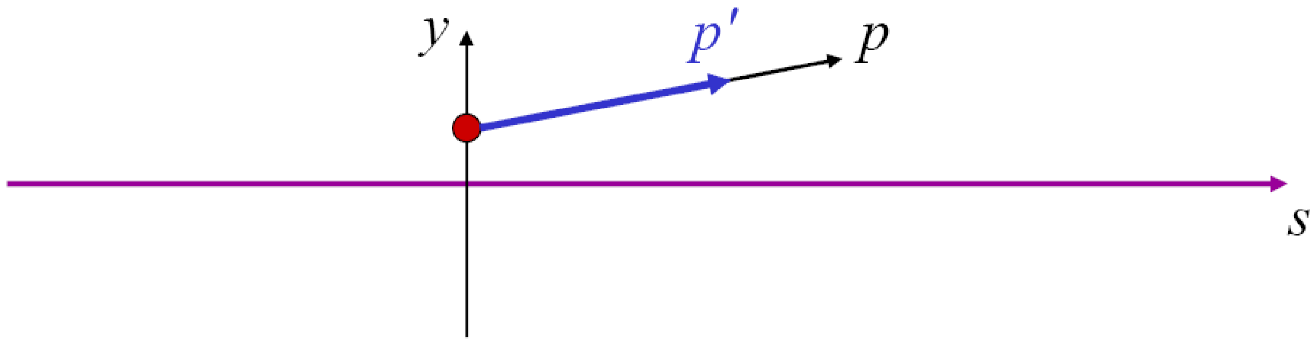
Radiation damping of vertical emittance



A relativistic particle will emit radiation with an opening angle of $1/\gamma$ with respect to its instantaneous direction of motion, where γ is the relativistic factor.

For an ultra-relativistic particle, $\gamma >> 1$, we can assume that the radiation is emitted directly along the instantaneous direction of motion of the particle.

Radiation damping of vertical emittance



The change in momentum of the particle is given by:

$$p' = p - dp \approx p\left(1 - \frac{dp}{P_0}\right)$$

where dp is the momentum carried by the radiation, and we assume that:

$$p \approx P_0$$

Since there is no change in direction of the particle, we must have:

$$p'_y \approx p_y\left(1 - \frac{dp}{P_0}\right)$$

Radiation damping of vertical emittance

After emission of radiation, the vertical momentum of the particle is:

$$p_y' = p_y\left(1 - \frac{dp}{P_0}\right)$$

Now we substitute this into the expression for the vertical betatron action (valid for *uncoupled* motion):

$$2J_y = \gamma_y y^2 + 2\alpha_y y p_y + \beta_y p_y^2$$

to find the change in the action resulting from the emission of radiation:

$$dJ_y = -\left(\alpha_y y p_y + \beta_y p_y^2\right)\frac{dp}{P_0}$$

We average over all particles in the beam, to find:

$$\langle dJ_y \rangle = d\varepsilon_y = -\varepsilon_y \frac{dp}{P_0}$$

where we have used: $\langle y p_y \rangle = -\alpha_y \varepsilon_y$ $\langle p_y^2 \rangle = \gamma_y \varepsilon_y$ and $\beta_y \gamma_y - \alpha_y^2 = 1$

Radiation damping of vertical emittance

For a particle moving round a storage ring, we can integrate the loss in momentum around the ring. The emittance is conserved under symplectic transport; so if the non-symplectic (radiation) effects are slow, we can write:

$$d\varepsilon_y = -\varepsilon_y \frac{dp}{P_0} \qquad \therefore \qquad \frac{d\varepsilon_y}{dt} = -\frac{\varepsilon_y}{T_0}\oint \frac{dp}{P_0} \approx -\frac{U_0}{E_0 T_0}\varepsilon_y$$

where T_0 is the revolution period, and U_0 is the energy loss in one turn. The approximation is valid for an ultra-relativistic particle, which has $E \approx pc$.

We define the damping time τ_y:

$$\boxed{\tau_y = 2\frac{E_0}{U_0}T_0}$$

so the evolution of the emittance is:

$$\boxed{\varepsilon_y(t) = \varepsilon_y(0)\exp\left(-2\frac{t}{\tau_y}\right)}$$

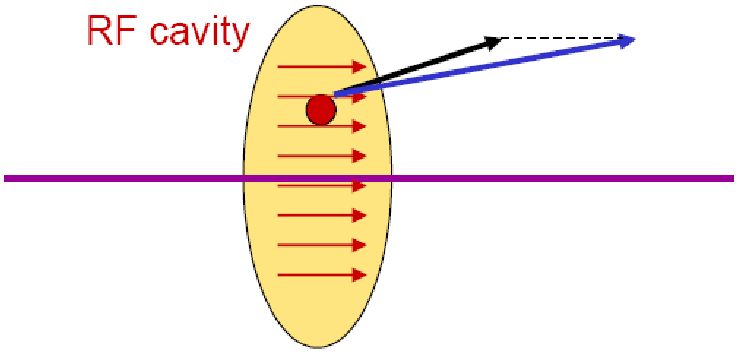
Typically, the damping time in a synchrotron storage ring is measured in tens of milliseconds, whereas the revolution period is measured in microseconds; so the radiation effects really are "slow".

Radiation damping of vertical emittance

Note that we made the assumption that the momentum of the particle was close to the reference momentum:

$$p \approx P_0$$

If the particle continues to radiate without any restoration of energy, we will reach a point where this assumption is no longer valid. However, electron storage rings contain RF cavities to restore the energy lost by synchrotron radiation. But then, we have to consider the change in momentum of a particle as it moves through an RF cavity.



Fortunately, RF cavities are usually designed with a longitudinal electric field, so that particles experience a change in longitudinal momentum as they pass through, without any change in transverse momentum.

Synchrotron radiation energy loss

To complete our calculation of the vertical damping time, we need to find the energy lost by a particle through synchrotron radiation on each turn through the storage ring. We quote the (classical) result that the power radiated by a particle of charge e and energy E in a magnetic field B is given by:

$$\boxed{P_\gamma = \frac{C_\gamma}{2\pi} c^3 e^2 B^2 E^2}$$

C_γ is a constant, given by:

$$\boxed{C_\gamma = \frac{e^2}{3\varepsilon_0 (mc^2)^4} \approx 8.846 \times 10^{-5} \ \mathrm{m/GeV^3}}$$

A charged particle with energy E in a magnetic field B follows a circular trajectory with radius ρ, given by:

$$B\rho = \frac{E}{ec}$$

Hence the synchrotron radiation power can be written:

$$\boxed{P_\gamma = \frac{C_\gamma}{2\pi} c \frac{E^4}{\rho^2}}$$

Synchrotron radiation energy loss

For a particle with the nominal energy, and traveling at (close to) the speed of light around the closed orbit, we can find the energy loss simply by integrating the radiation power around the ring:

$$U_0 = \oint P_\gamma \, dt = \oint P_\gamma \frac{ds}{c}$$

Using the previous expression for P_γ, we find:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 \oint \frac{1}{\rho^2} \, ds$$

Conventionally, we define the *second synchrotron radiation integral*, I_2:

$$\boxed{I_2 = \oint \frac{1}{\rho^2} \, ds}$$

In terms of I_2, the energy loss per turn U_0 is written:

$$\boxed{U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2}$$

The first synchrotron radiation integral

Note that I_2 is a property of the lattice (actually, of the reference trajectory), and does not depend on the properties of the beam.

Conventionally, there are five synchrotron radiation integrals defined, which are used to express in convenient form the dynamics of a beam emitting radiation.

The first synchrotron radiation integral is not, however, directly related to the radiation effects. It is defined as:

$$I_1 = \oint \frac{\eta_x}{\rho} ds$$

where η_x is the horizontal dispersion.

The momentum compaction factor, α_p, can be written:

$$\alpha_p \equiv \frac{1}{C_0} \frac{dC}{d\delta}\bigg|_{\delta=0} = \frac{1}{C_0} \oint \frac{\eta_x}{\rho} ds = \frac{1}{C_0} I_1$$

Damping of horizontal emittance

Analysis of the radiation effects on the vertical emittance was relatively straightforward. When we consider the horizontal emittance, there are three complications that we need to address:

- The horizontal motion of a particle is often strongly coupled to the longitudinal motion.
- Where the reference trajectory is curved (usually, in dipoles), the path length taken by a particle depends on the horizontal coordinate with respect to the reference trajectory.
- Dipole magnets are sometimes built with a gradient, so that the vertical field seen by a particle in a dipole depends on the horizontal coordinate of the particle.

Horizontal-longitudinal coupling

Coupling between transverse and longitudinal planes in a beam line is usually represented by the dispersion, η_x. So, in terms of the horizontal dispersion, the horizontal coordinate and momentum of a particle are given by:

$$x = \sqrt{2\beta_x J_x}\cos\varphi_x + \eta_x \delta$$

$$p_x = -\sqrt{\frac{2J_x}{\beta_x}}\left(\sin\varphi_x + \alpha_x \cos\varphi_x\right) + \eta_{px}\delta$$

When a particle emits radiation, we have to take into account:

- the change in momentum of the particle (because of the momentum carried by the radiation);
- the change in coordinate x and momentum p_x resulting from the change in energy deviation δ.

When we analysed the vertical motion, we ignored the second effect, because we assumed that the vertical dispersion was zero.

Damping of horizontal emittance

Taking all the above effects into account, we can proceed along the same lines as for the analysis of the vertical emittance. That is:

- Write down the changes in coordinate x and momentum p_x resulting from an emission of radiation with momentum dp (taking into account the additional effects of dispersion).
- Substitute expressions for the new coordinate and momentum into the expression for the horizontal betatron action, to find the change in action resulting from the radiation emission.
- Average over all particles in the beam, to find the change in the emittance resulting from radiation emission from each particle.
- Integrate around the ring (taking account of changes in path length and field strength with x in the bends) to find the change in emittance over one turn.

The algebra gets somewhat cumbersome, and is not especially enlightening: see Appendix A for more details. Here, we just quote the result…

Radiation damping of horizontal emittance

The horizontal emittance decays exponentially:

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x$$

where the horizontal damping time is given by:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0}T_0$$

The horizontal damping partition number j_x is given by:

$$j_x = 1 - \frac{I_4}{I_2}$$

where the fourth synchrotron radiation integral I_4 is given by:

$$I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right) ds \qquad k_1 = \frac{e}{P_0}\frac{\partial B_y}{\partial x}$$

Appendix A: Damping of horizontal emittance

In this Appendix, we derive the expression for radiation damping of the horizontal emittance:

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x$$

where:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0}T_0 \qquad\qquad j_x = 1 - \frac{I_4}{I_2}$$

To do this, we proceed as follows:

1. We find an expression for the change of horizontal action of a single particle when emitting radiation with momentum dp.
2. We integrate around the ring to find the change in action per revolution period.
3. We average the action over all particles in the bunch, to find the change in emittance per revolution period.

Appendix A: Damping of horizontal emittance

To begin, we note that, in the presence of dispersion, the action J_x is written:

$$2J_x = \gamma_x \tilde{x}^2 + 2\alpha_x \tilde{x}\tilde{p}_x + \beta_x \tilde{p}_x^2$$

where:

$$\tilde{x} = x - \eta_x \delta \qquad \tilde{p}_x = p_x - \eta_{px} \delta$$

After emission of radiation carrying momentum dp, the variables change by:

$$\delta \mapsto \delta - \frac{dp}{P_0} \qquad \tilde{x} \mapsto \tilde{x} + \eta_x \frac{dp}{P_0} \qquad \tilde{p}_x \mapsto \tilde{p}_x \left(1 - \frac{dp}{P_0}\right) + \eta_{px}(1-\delta)\frac{dp}{P_0}$$

The resulting change in the action is:

$$J_x \mapsto J_x + dJ_x$$

Appendix A: Damping of horizontal emittance

The change in the horizontal action is:

$$dJ_x = -w_1 \frac{dp}{P_0} + w_2 \left(\frac{dp}{P_0} \right)^2 \qquad \therefore \qquad \frac{dJ_x}{dt} = -w_1 \frac{1}{P_0} \frac{dp}{dt} + w_2 \frac{dp}{P_0^2} \frac{dp}{dt} \tag{A1}$$

where, in the limit $\delta \to 0$:

$$w_1 = \alpha_x x p_x + \beta_x p_x^2 - \eta_x (\gamma_x x + \alpha_x p_x) - \eta_{px} (\alpha_x x + \beta_x p_x) \tag{A2}$$

and:

$$w_2 = \frac{1}{2} (\gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2) - (\alpha_x \eta_x + \beta_x \eta_{px}) p_x + \frac{1}{2} \beta_x p_x^2 \tag{A3}$$

Treating radiation as a classical phenomenon, we can take the limit $dp \to 0$ in the limit of small time interval, $dt \to 0$. In this approximation:

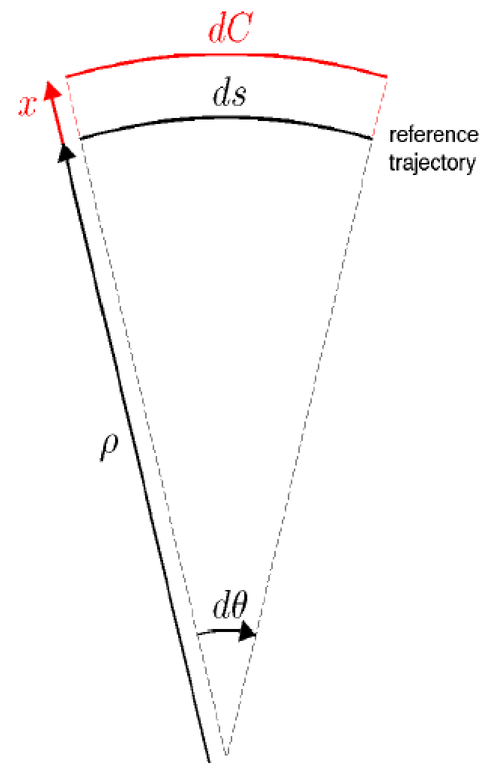
$$\frac{dJ_x}{dt} \approx -w_1 \frac{1}{P_0} \frac{dp}{dt} \approx -w_1 \frac{P_\gamma}{P_0 c}$$

where P_γ is the *rate of energy loss* of the particle through radiation.

To find the *average* rate of change of horizontal action, we integrate over one revolution period:

$$\frac{dJ_x}{dt} = -\frac{1}{T_0}\oint w_1 \frac{P_\gamma}{P_0 c}\,dt$$

We have to be careful changing the variable of integration where the reference trajectory is curved:



$$dt = \frac{dC}{c} = \left(1 + \frac{x}{\rho}\right)\frac{ds}{c}$$

So:

$$\frac{dJ_x}{dt} = -\frac{1}{T_0 P_0 c^2}\oint w_1 P_\gamma \left(1 + \frac{x}{\rho}\right) ds \qquad \text{(A4)}$$

where the rate of energy loss is:

$$P_\gamma = \frac{C_\gamma}{2\pi} c^3 e^2 B^2 E^2 \qquad \text{(A5)}$$

Appendix A: Damping of horizontal emittance

We have to take into account the fact that the field strength in a dipole can vary with position. To first order in x we can write:

$$B = B_0 + x\frac{\partial B_y}{\partial x} \qquad \text{(A6)}$$

Substituting equation (A6) into (A5), and with the use of (A2), we find (after some algebra!) that, averaging over all particles in the beam:

$$\oint \left\langle w_1 P_\gamma \left(1+\frac{x}{\rho}\right)\right\rangle ds = cU_0\left(1-\frac{I_4}{I_2}\right)\varepsilon_x \qquad \text{(A7)}$$

where:

$$U_0 = \frac{C_\gamma}{2\pi}cE_0^4 I_2 \qquad I_2 = \oint \frac{1}{\rho^2}ds \qquad I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2}+2k_1\right)ds$$

and k_1 is the quadrupole gradient in the dipole field:

$$k_1 = \frac{e}{P_0}\frac{\partial B_y}{\partial x}$$

Appendix A: Damping of horizontal emittance

Combining equations (A4) and (A7) we have:

$$\frac{d\varepsilon_x}{dt} = -\frac{1}{T_0}\frac{U_0}{E_0}\left(1-\frac{I_4}{I_2}\right)\varepsilon_x$$

Defining the horizontal damping time, τ_x:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0}T_0 \qquad\qquad j_x = 1-\frac{I_4}{I_2}$$

the evolution of the horizontal emittance can be written:

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x$$

The quantity j_x is called the horizontal damping partition number. For most lattices, if there is no gradient in the dipoles, then j_x is very close to 1.

Damping of synchrotron oscillations

So far, we have considered the effects of synchrotron radiation on the transverse motion. There are also effects on the longitudinal motion.

Generally, synchrotron oscillations are handled differently from betatron oscillations, because the synchrotron tune in a storage ring is usually much less than 1, whereas the betatron tunes are much greater than 1.

To find the effects of radiation on synchrotron motion, we proceed as follows:

- We write down the equations of motion (for the variables z and δ) for a particle performing synchrotron motion, including the radiation energy loss.
- We express the energy loss per turn as a function of the energy deviation of the particle. This introduces a "damping term" into the equations of motion.
- Solving the equations of motion gives synchrotron oscillations (as expected) with amplitude that decays exponentially.

Damping of synchrotron oscillations

The change in energy deviation δ and longitudinal coordinate z for a particle in one turn around a storage ring are given by:

$$\Delta\delta = \frac{eV_{RF}}{E_0}\sin\left(\varphi_s - \frac{\omega_{RF} z}{c}\right) - \frac{U}{E_0}$$

$$\Delta z = -\alpha_p C_0 \delta$$

where V_{RF} is the RF voltage and ω_{RF} the RF frequency, E_0 is the reference energy of the beam, φ_s is the nominal RF phase, and U is the energy lost by the particle through synchrotron radiation.

If the revolution period is T_0, then we can write the longitudinal equations of motion for the particle:

$$\boxed{\begin{aligned} \frac{d\delta}{dt} &= \frac{eV_{RF}}{E_0 T_0}\sin\left(\varphi_s - \frac{\omega_{RF} z}{c}\right) - \frac{U}{E_0 T_0} \\ \frac{dz}{dt} &= -\alpha_p c \delta \end{aligned}}$$

Damping of synchrotron oscillations

Let us assume that z is small compared to the RF wavelength, i.e. $\omega_{RF} z/c << 1$.

Also, the energy loss per turn is a function of the energy of the particle (particles with higher energy radiate higher synchrotron radiation power), so we can write (to first order in the energy deviation):

$$U = U_0 + \Delta E \left.\frac{dU}{dE}\right|_{E=E_0} = U_0 + E_0 \delta \left.\frac{dU}{dE}\right|_{E=E_0}$$

Further, we assume that the RF phase φ_s is set so that for $z = \delta = 0$, the RF cavity restores exactly the amount of energy lost by synchrotron radiation. The equations of motion then become:

$$\frac{d\delta}{dt} = -\frac{eV_{RF}}{E_0 T_0} \cos\varphi_s \frac{\omega_{RF}}{c} z - \frac{1}{T_0} \delta \left.\frac{dU}{dE}\right|_{E=E_0}$$

$$\frac{dz}{dt} = -\alpha_p c \delta$$

Damping of synchrotron oscillations

Combining these equations gives:

$$\frac{d^2\delta}{dt^2} + 2\alpha_E \frac{d\delta}{dt} + \omega_s^2 \delta = 0$$

This is the equation for a damped harmonic oscillator, with frequency ω_s and damping constant α_E given by:

$$\omega_s^2 = -\frac{eV_{RF}}{E_0} \cos\varphi_s \frac{\omega_{RF}}{T_0} \alpha_p$$

$$\alpha_E = \frac{1}{2T_0} \left. \frac{dU}{dE} \right|_{E=E_0}$$

Damping of synchrotron oscillations

If $\alpha_E << \omega_s$, the energy deviation and longitudinal coordinate damp as:

$$\delta(t) = \hat{\delta}\exp(-\alpha_E t)\sin(\omega_s t - \theta_0)$$

$$z(t) = \frac{\alpha_p c}{\omega_s}\hat{\delta}\exp(-\alpha_E t)\cos(\omega_s t - \theta_0)$$

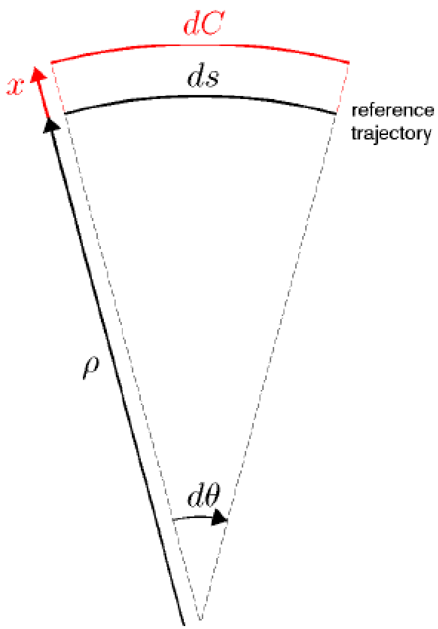
To find the damping constant α_E, we need to know how the energy loss per turn U depends on the energy deviation δ…

Damping of synchrotron oscillations

We can find the total energy lost by integrating over one revolution period:

$$U = \oint P_\gamma \, dt$$

To convert this to an integral over the circumference, we should recall that the path length depends on the energy deviation; so a particle with a higher energy takes longer to travel round the lattice.



$$dt = \frac{dC}{c}$$

$$dC = \left(1 + \frac{x}{\rho}\right) ds = \left(1 + \frac{\eta_x \delta}{\rho}\right) ds$$

$$U = \frac{1}{c} \oint P_\gamma \left(1 + \frac{\eta_x \delta}{\rho}\right) ds$$

Damping of synchrotron oscillations

With the energy loss per turn given by:

$$U = \frac{1}{c}\oint P_\gamma \left(1 + \frac{\eta_x}{\rho}\delta\right) ds$$

and the synchrotron radiation power given by:

$$P_\gamma = \frac{C_\gamma}{2\pi} c^3 e^2 B^2 E^2 = \frac{C_\gamma}{2\pi} c \frac{E^4}{\rho^2}$$

we find, after some algebra:

$$\left.\frac{dU}{dE}\right|_{E=E_0} = j_E \frac{U_0}{E_0}$$

where:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2 \qquad j_E = 2 + \frac{I_4}{I_2}$$

I_2 and I_4 are the same synchrotron radiation integrals that we saw before:

$$I_2 = \oint \frac{1}{\rho^2} ds \qquad I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right) ds \qquad k_1 = \frac{e}{P_0}\frac{\partial B_y}{\partial x}$$

Damping of synchrotron oscillations

Finally, we can write the longitudinal damping time:

$$\boxed{\tau_z = \frac{1}{\alpha_E} = \frac{2}{j_z}\frac{E_0}{U_0}T_0}$$

U_0 is the energy loss per turn for a particle with the reference energy E_0, following the reference trajectory. It is given by:

$$U_0 = \frac{C_\gamma}{2\pi}E_0^4 I_2$$

j_z is the longitudinal damping partition number, given by:

$$\boxed{j_z = 2 + \frac{I_4}{I_2}}$$

Damping of synchrotron oscillations

The longitudinal emittance is given by a similar expression to the horizontal and vertical emittances:

$$\varepsilon_z = \sqrt{\langle z^2 \rangle \langle \delta^2 \rangle - \langle z\delta \rangle^2}$$

In most storage rings, the correlation $\langle z\delta \rangle$ is negligible, so the emittance becomes:

$$\varepsilon_z \approx \sigma_z \sigma_\delta$$

Hence, the damping of the longitudinal emittance can be written:

$$\varepsilon_z(t) = \varepsilon_z(0) \exp\left(-2\frac{t}{\tau_z} \right)$$

Summary: synchrotron radiation damping

The energy loss per turn is given by:

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 I_2 \qquad C_\gamma = 8.846\times 10^{-5}\ \text{m/GeV}^3$$

The radiation damping times are given by:

$$\tau_x = \frac{2}{j_x}\frac{E_0}{U_0}T_0 \qquad \tau_y = \frac{2}{j_y}\frac{E_0}{U_0}T_0 \qquad \tau_z = \frac{2}{j_z}\frac{E_0}{U_0}T_0$$

The damping partition numbers are:

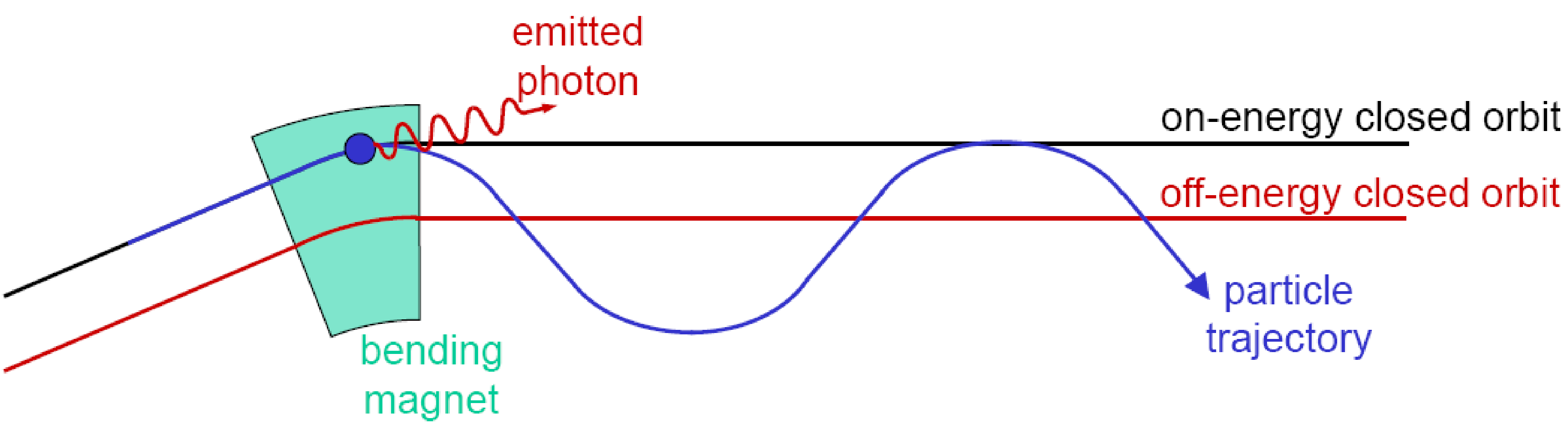
$$j_x = 1 - \frac{I_4}{I_2} \qquad j_y = 1 \qquad j_z = 2 + \frac{I_4}{I_2}$$

The second and fourth synchrotron radiation integrals are:

$$I_2 = \oint \frac{1}{\rho^2}ds \qquad I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right)ds$$

Quantum excitation

If radiation were a purely classical process, the emittances would damp to nearly zero. However radiation is emitted in discrete units (photons), which induces some "noise" on the beam. The effect of the noise is to increase the emittance. The beam eventually reaches an equilibrium determined by a balance between the radiation damping and the quantum excitation.



Quantum excitation of horizontal emittance

By considering the change in the phase-space variables when a particle emits radiation carrying momentum dp, we find that the associated change in the betatron action is:

$$dJ_x = -w_1 \frac{dp}{P_0} + w_2 \left(\frac{dp}{P_0} \right)^2$$

where w_1 and w_2 are functions of the Twiss parameters, the dispersion, and the phase-space variables (see Appendix A).

The time evolution of the action can then be written:

$$\frac{dJ_x}{dt} = -w_1 \frac{1}{P_0} \frac{dp}{dt} + w_2 \frac{dp}{P_0^2} \frac{dp}{dt}$$

In the classical approximation, we can take $dp \to 0$ in the limit of small time interval, $dt \to 0$. In this approximation, the second term on the right hand side in the above equation vanishes, and we are left only with damping. But since radiation is quantized, it makes no real sense to take $dp \to 0$…

Quantum excitation of horizontal emittance

To take account of the quantization of synchrotron radiation, we write the time-evolution of the action as:

$$\frac{dJ_x}{dt} = -w_1 \frac{1}{P_0}\frac{dp}{dt} + w_2 \frac{dp}{P_0^2}\frac{dp}{dt} \qquad \therefore \qquad \frac{dJ_x}{dt} = -w_1 \dot{N}\frac{\langle u \rangle}{P_0 c} + w_2 \dot{N}\frac{\langle u^2 \rangle}{P_0^2 c^2}$$

where u is the photon energy, and $\dot{N}$ is the number of photons emitted per unit time.

In Appendix B, we show that this leads to the equation for the evolution of the emittance, including both radiation damping and quantum excitation:

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x + \frac{2}{j_x \tau_x} C_q \gamma^2 \frac{I_5}{I_2}$$

where the fifth synchrotron radiation integral I_5 is given by:

$$\boxed{I_5 = \oint \frac{\mathcal{H}_x}{|\rho^3|} ds}$$

and the "quantum constant" C_q is given by: $C_q = \frac{55}{32\sqrt{3}}\frac{\hbar}{mc} \approx 3.832 \times 10^{-13}$ m

Appendix B: Quantum excitation of horizontal emittance

In deriving the equation of motion (A4) for the action of a particle emitting synchrotron radiation, we made the classical approximation that in a time interval dt, the momentum of the radiation emitted dp goes to zero as dt goes to zero.

In reality, emission of radiation is quantized, so writing "$dp \to 0$" actually makes no sense.

Taking into account the quantization of radiation, the equation of motion for the action (A1) should be written:

$$dJ_x = -w_1 \frac{dp}{P_0} + w_2 \left(\frac{dp}{P_0} \right)^2 \qquad \therefore \qquad \frac{dJ_x}{dt} = -w_1 \dot{N} \frac{\langle u \rangle}{P_0 c} + w_2 \dot{N} \frac{\langle u^2 \rangle}{P_0^2 c^2} \qquad \text{(B1)}$$

where $\dot{N}$ is the number of photons emitted per unit time.

The first term on the right hand side of (B1) just gives the same radiation damping as in the classical approximation. The second term on the right hand side of (B1) is an excitation term that we previously neglected…

Appendix B: Quantum excitation of horizontal emittance

Averaging around the circumference of the ring, the quantum excitation term can be written:

$$w_2 \dot{N} \frac{\langle u^2 \rangle}{P_0^2 c^2} \approx \frac{1}{C_0} \oint w_2 \dot{N} \frac{\langle u^2 \rangle}{P_0^2 c^2} ds$$

Using equation (A3) for w_2, we find that (for $x \ll \eta_x$ and $p_x \ll \eta_{px}$) the excitation term can be written:

$$w_2 \dot{N} \frac{\langle u^2 \rangle}{P_0^2 c^2} \approx \frac{1}{2E_0^2 C_0} \oint \mathcal{H}_x \dot{N} \langle u^2 \rangle ds$$

where the "curly-H" function $\mathcal{H}_x$ is given by:

$$\mathcal{H}_x = \gamma_x \eta_x{}^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2$$

Appendix B: Quantum excitation of horizontal emittance

Including both (classical) damping and (quantum) excitation terms, and averaging over all particles in the bunch, we find that the horizontal emittance evolves as:

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x + \frac{1}{2E_0^2 C_0}\oint \dot{N}\left\langle u^2 \right\rangle \mathcal{H}_x \, ds \qquad \text{(B2)}$$

We quote the result (from quantum radiation theory):

$$\dot{N}\left\langle u^2 \right\rangle = 2C_q \gamma^2 E_0 \frac{P_\gamma}{\rho} \qquad \text{(B3)}$$

where the "quantum constant" C_q is:

$$C_q = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} \approx 3.832 \times 10^{-13} \text{ m}$$

Appendix B: Quantum excitation of horizontal emittance

Using equation (B3), and equation (A5) for P_γ, and the results:

$$j_x \tau_x = 2\frac{E_0}{U_0} T_0 \qquad U_0 = \frac{C_\gamma}{2\pi} c E_0^4 I_2$$

we find that equation (B2) for the evolution of the emittance can be written:

$$\frac{d\varepsilon_x}{dt} = -\frac{2}{\tau_x}\varepsilon_x + \frac{2}{j_x \tau_x} C_q \gamma^2 \frac{I_5}{I_2}$$

where the fifth synchrotron radiation integral I_5 is given by:

$$I_5 = \oint \frac{\mathcal{H}_x}{|\rho^3|} ds$$

Note that the excitation term is independent of the emittance: it does not simply modify the damping time, but leads to a non-zero equilibrium emittance.

Quantum excitation of horizontal emittance

The equilibrium horizontal emittance is found from:

$$\left.\frac{d\varepsilon_x}{dt}\right|_{\varepsilon_x=\varepsilon_0} = 0 \qquad \therefore \qquad \frac{2}{\tau_x}\varepsilon_0 = \frac{2}{j_x\tau_x} C_q\gamma^2 \frac{I_5}{I_2}$$

The equilibrium horizontal emittance is given by:

$$\boxed{\varepsilon_0 = C_q\gamma^2 \frac{I_5}{j_x I_2}}$$

Note that ε_0 is determined by the beam energy, the lattice functions (Twiss parameters and dispersion) in the dipoles, and the bending radius in the dipoles.

ε_0 is sometimes called the "natural emittance" of the lattice, since it is the horizontal emittance that will be achieved in the limit of zero bunch charge: as the current is increased, interactions betweens particles in a bunch can increase the emittance above the equilibrium determined by radiation effects.

Quantum excitation of vertical emittance

In many storage rings, the vertical dispersion in the absence of alignment, steering and coupling errors is zero, so $\mathcal{H}_y = 0$. However, the equilibrium vertical emittance is larger than zero, because the vertical opening angle of the radiation excites some vertical betatron oscillations.

The fundamental lower limit on the vertical emittance, from the opening angle of the synchrotron radiation, is given by[1]:

$$\varepsilon_y = \frac{13}{55} \frac{C_q}{j_y I_2} \oint \frac{\beta_y}{|\rho^3|} ds$$

In most storage rings, this is an extremely small value, typically four orders of magnitude smaller than the natural (horizontal) emittance.

In practice, the vertical emittance is dominated by magnet alignment errors. Storage rings typically operate with a vertical emittance that is of order 1% of the horizontal emittance, but many can achieve emittance ratios somewhat smaller than this.

[1] T. Raubenheimer, SLAC Report 387, p.19 (1991).

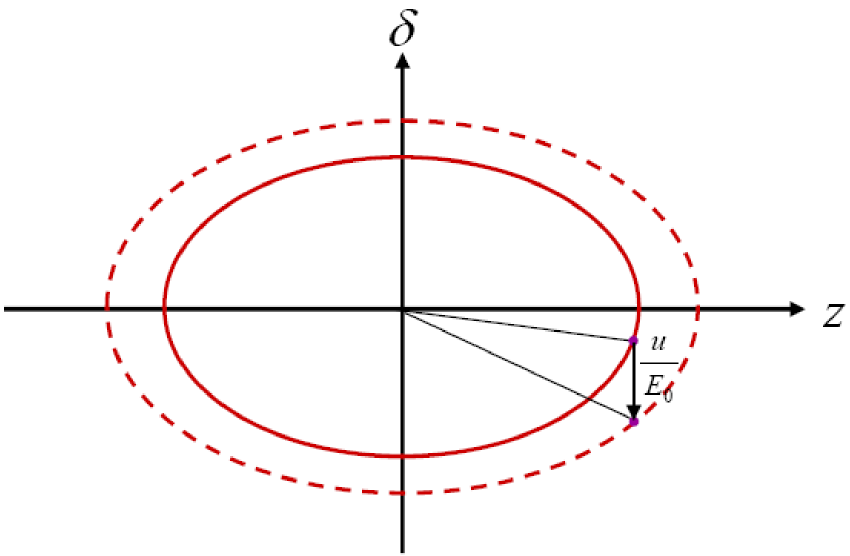
Quantum excitation of synchrotron oscillations

Quantum effects excite longitudinal emittance as well as transverse emittance. Consider a particle with longitudinal coordinate z and energy deviation δ, which emits a photon of energy u.

$$\delta' = \hat{\delta}' \sin\theta' = \hat{\delta}\sin\theta - \frac{u}{E_0}$$

$$z' = \frac{\alpha_p c}{\omega_s}\hat{\delta}'\cos\theta' = \frac{\alpha_p c}{\omega_s}\hat{\delta}\cos\theta$$

$$\therefore \quad \hat{\delta}'^2 = \hat{\delta}^2 - 2\hat{\delta}\frac{u}{E_0}\sin\theta + \frac{u^2}{E_0^2}$$



Averaging over the bunch gives:

$$\Delta\sigma_\delta^2 = \frac{\langle u^2 \rangle}{2E_0^2} \qquad \text{where} \qquad \sigma_\delta^2 = \frac{1}{2}\langle \hat{\delta}^2 \rangle$$

Quantum excitation of synchrotron oscillations

Including the effects of radiation damping, the evolution of the energy spread is:

$$\frac{d\sigma_\delta^2}{dt} = \frac{1}{2E_0^2 C_0} \oint \dot{N}\left\langle u^2 \right\rangle ds - \frac{2}{\tau_z}\sigma_\delta^2$$

Using equation (B3) from Appendix B for $\dot{N}\langle u^2\rangle$, we find:

$$\frac{d\sigma_\delta^2}{dt} = C_q \gamma^2 \frac{2}{j_z \tau_z}\frac{I_3}{I_2} - \frac{2}{\tau_z}\sigma_\delta^2$$

We find the equilibrium energy spread from $d\sigma_\delta^2/dt = 0$:

$$\boxed{\sigma_{\delta 0}^2 = C_q \gamma^2 \frac{I_3}{j_z I_2}}$$

The third synchrotron radiation integral I_3 is given by:

$$\boxed{I_3 = \oint \frac{1}{\left|\rho^3\right|} ds}$$

Natural energy spread

The equilibrium energy spread determined by radiation effects is:

$$\sigma_{\delta 0}^2 = C_q \gamma^2 \frac{I_3}{j_z I_2}$$

This is often referred to as the "natural" energy spread, since collective effects can often lead to an increase in the energy spread with increasing bunch charge.

The natural energy spread is determined essentially by the beam energy and by the bending radii of the dipoles. Note that the natural energy spread *does not depend on the RF parameters (either voltage or frequency)*.

The corresponding equilibrium bunch length is:

$$\sigma_z = \frac{\alpha_p c}{\omega_s} \sigma_\delta$$

We can increase the synchrotron frequency ω_s, and hence reduce the bunch length, by increasing the RF voltage, or by increasing the RF frequency.

Summary: radiation damping

Including the effects of radiation damping and quantum excitation, the emittances vary as:

$$\varepsilon(t) = \varepsilon(0)\exp\left(-\frac{2t}{\tau}\right) + \varepsilon(\infty)\left[1 - \exp\left(-\frac{2t}{\tau}\right)\right]$$

The damping times are given by:

$$j_x\tau_x = j_y\tau_y = j_z\tau_z = 2\frac{E_0}{U_0}T_0$$

The damping partition numbers are given by:

$$j_x = 1 - \frac{I_4}{I_2} \qquad j_y = 1 \qquad j_z = 2 + \frac{I_4}{I_2}$$

The energy loss per turn is given by:

$$U_0 = \frac{C_\gamma}{2\pi}E_0^4 I_2 \qquad C_\gamma = 8.846\times10^{-5}\ \text{m/GeV}^3$$

Summary: synchrotron radiation integrals

The natural emittance is:

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{j_x I_2} \qquad C_q = 3.832 \times 10^{-13}\ \text{m}$$

The natural energy spread and bunch length are given by:

$$\sigma_\delta^2 = C_q \gamma^2 \frac{I_3}{j_z I_2} \qquad \sigma_z = \frac{\alpha_p c}{\omega_s} \sigma_\delta$$

The momentum compaction factor is:

$$\alpha_p = \frac{I_1}{C_0}$$

The synchrotron frequency and synchronous phase are given by:

$$\omega_s^2 = -\frac{eV_{RF}}{E_0} \frac{\omega_{RF}}{T_0} \alpha_p \cos\varphi_s \qquad \sin\varphi_s = \frac{U_0}{eV_{RF}}$$

Summary: synchrotron radiation integrals

The synchrotron radiation integrals are:

$$I_1 = \oint \frac{\eta_x}{\rho}\, ds$$

$$I_2 = \oint \frac{1}{\rho^2}\, ds$$

$$I_3 = \oint \frac{1}{|\rho|^3}\, ds$$

$$I_4 = \oint \frac{\eta_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right) ds \qquad k_1 = \frac{e}{P_0}\frac{\partial B_y}{\partial x}$$

$$I_5 = \oint \frac{\mathcal{H}_x}{|\rho|^3}\, ds \qquad \mathcal{H}_x = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2$$

Radiation Damping - Re-cap

- So far we:
  - discussed the effect of synchrotron radiation on the (linear) motion of particles in storage rings;
  - derived expressions for the damping times of the vertical, horizontal and longitudinal emittances;
  - discussed the effects of quantum excitation, and derive expressions for the equilibrium horizontal and longitudinal beam emittances in an electron storage ring.

S-R integrals - Re-cap

The natural emittance is:

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{j_x I_2} \qquad C_q = 3.832 \times 10^{-13} \text{ m}$$

The natural energy spread and bunch length are given by:

$$\sigma_\delta^2 = C_q \gamma^2 \frac{I_3}{j_z I_2} \qquad \sigma_z = \frac{\alpha_p c}{\omega_s} \sigma_\delta$$

The momentum compaction factor is:

$$\alpha_p = \frac{I_1}{C_0}$$

The synchrotron frequency and synchronous phase are given by:

$$\omega_s^2 = -\frac{eV_{RF}}{E_0} \frac{\omega_{RF}}{T_0} \alpha_p \cos\varphi_s \qquad \sin\varphi_s = \frac{U_0}{eV_{RF}}$$

S-R integrals - Re-cap

The synchrotron radiation integrals are:

$$I_1 = \oint \frac{\eta_x}{\rho} ds$$

$$I_2 = \oint \frac{1}{\rho^2} ds$$

$$I_3 = \oint \frac{1}{|\rho|^3} ds$$

$$I_4 = \oint \frac{\eta_x}{\rho} \left(\frac{1}{\rho^2} + 2k_1 \right) ds \qquad k_1 = \frac{e}{P_0} \frac{\partial B_y}{\partial x}$$

$$I_5 = \oint \frac{\mathcal{H}_x}{|\rho|^3} ds \qquad \mathcal{H}_x = \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta_{px} + \beta_x \eta_{px}^2$$

Lattice Examples

- Practical implementations:
  - FODO
  - DBA (double-bend achromat)
  - multi-bend achromat, including the triple-bend achromat (TBA)
  - TME (theoretical minimum emittance)

Calculating the natural emittance in a lattice

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{j_x I_2}$$

where C_q is a physical constant, γ is the relativistic factor, j_x is the horizontal damping partition number, and I_5 and I_2 are synchrotron radiation integrals

j_x, I_5 and I_2 are all functions of the lattice, and independent of the beam energy.

In most storage rings, if the bends have no quadrupole component, the damping partition number $j_x \approx 1$. In this case, we just need to evaluate the two synchrotron radiation integrals:

$$I_5 = \int \frac{\mathcal{H}_x}{\rho^3} ds \qquad I_2 = \int \frac{1}{\rho^2} ds$$

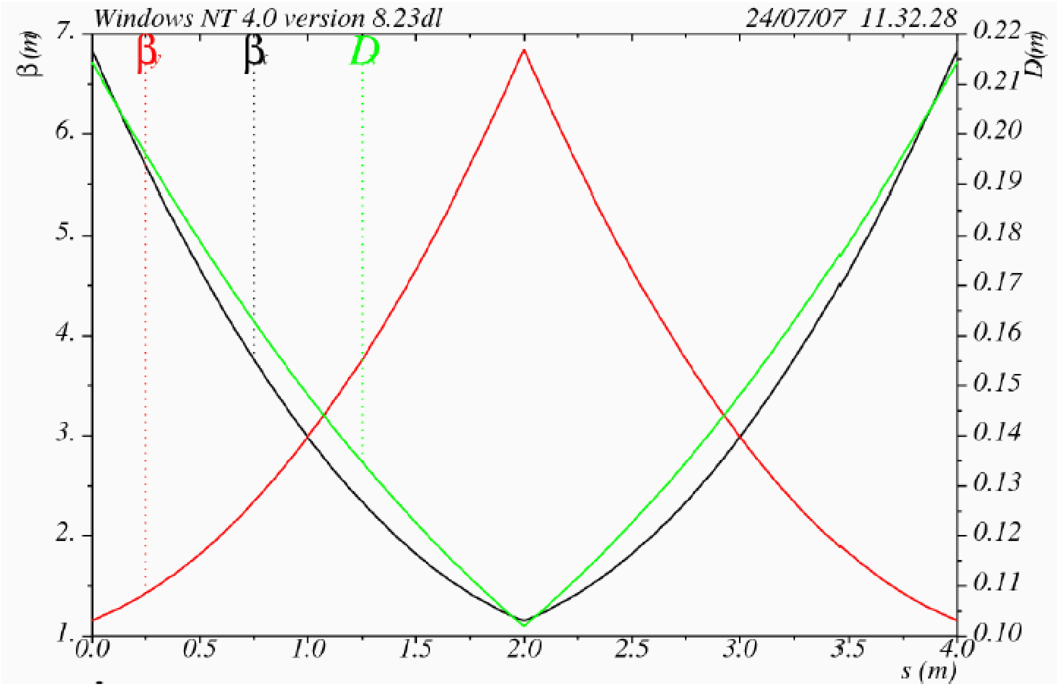
If we know the strength and length of all the dipoles in the lattice, it is straightforward to evaluate I_2.

Evaluating I_5 is more complicated: it depends on the lattice functions…

FODO lattice - natural emittance

Let us consider the case of a simple FODO lattice. To simplify this case, we will use the following approximations:

- the quadrupoles are represented as thin lenses;
- the space between the quadrupoles is completely filled by the dipoles.



FODO lattice - natural emittance

With the approximations in the previous slide, the lattice functions (Twiss parameters and dispersion) are completely determined by the following parameters:

- the focal length f of a quadrupole;
- the bending radius ρ of a dipole;
- the length L of a dipole.

The bending angle θ of a dipole is given by: $\theta = \frac{L}{\rho}$

In terms of these parameters, the horizontal beta function and dispersion at the centre of the horizontally-focusing quadrupole are given by:

$$\beta_x = \frac{4f\rho\sin\theta(2f\cos\theta + \rho\sin\theta)}{\sqrt{16f^4 - \left[\rho^2 - (4f^2+\rho^2)\cos 2\theta\right]^2}} \qquad \eta_x = \frac{2f\rho\left(2f + \rho\tan\frac{\theta}{2}\right)}{4f^2+\rho^2}$$

By symmetry, at the centre of a quadrupole, $\alpha_x = \eta_{px} = 0$.

FODO lattice - natural emittance

We also know how to evolve the lattice functions through the lattice, using the transfer matrices, M.

For the Twiss parameters, we use: $A(s) = M \cdot A(0) \cdot M^{\mathrm{T}}$

where: $$A = \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix}$$

The dispersion can be evolved using: $$\begin{pmatrix} \eta_x \\ \eta_{px} \end{pmatrix}_s = M \cdot \begin{pmatrix} \eta_x \\ \eta_{px} \end{pmatrix}_{s=0} + \begin{pmatrix} \rho\left(1-\cos\frac{s}{\rho}\right) \\ \sin\frac{s}{\rho} \end{pmatrix}$$

For a thin quadrupole, the transfer matrix is given by: $$M = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

For a dipole, the transfer matrix is given by: $$M = \begin{pmatrix} \cos\frac{s}{\rho} & \rho\sin\frac{s}{\rho} \\ -\frac{1}{\rho}\sin\frac{s}{\rho} & \cos\frac{s}{\rho} \end{pmatrix}$$

FODO lattice - natural emittance

With the expressions for the Twiss parameters and dispersion from the previous two slides, we can evaluate the synchrotron radiation integral I_5.

Note: by symmetry, we need to evaluate the integral in only one of the two dipoles in the FODO cell.

The algebra is rather formidable. The result is most easily expressed as a power series in the dipole bending angle θ. We find that:

$$\frac{I_5}{I_2} = \left(4 + \frac{\rho^2}{f^2}\right)^{-\frac{3}{2}} \left[8 - \frac{\rho^2}{2f^2}\theta^2 + O\left(\theta^4\right)\right]$$

FODO lattice - natural emittance

For small θ, the expression for I_5/I_2 can be written:

$$\frac{I_5}{I_2} \approx \left(1 - \frac{\rho^2}{16 f^2}\theta^2\right)\left(1 + \frac{\rho^2}{4 f^2}\right)^{-\frac{3}{2}} = \left(1 - \frac{L^2}{16 f^2}\right)\left(1 + \frac{\rho^2}{4 f^2}\right)^{-\frac{3}{2}}$$

This can be further simplified if $\rho >> 2f$ (which is often the case):

$$\frac{I_5}{I_2} \approx \left(1 - \frac{L^2}{16 f^2}\right)\frac{8 f^3}{\rho^3}$$

and still further if $4f >> L$ (which is less generally the case):

$$\frac{I_5}{I_2} \approx 8\frac{f^3}{\rho^3}$$

Making the approximation $j_x \approx 1$ (since we have no quadrupole component in the dipole), and writing $\rho = L/\theta$, we have:

$$\varepsilon_0 \approx C_q \gamma^2 \left(\frac{2f}{L}\right)^3 \theta^3 \qquad\qquad \rho >> 2f >> L/2$$

FODO lattice - natural emittance

We have derived an approximate expression for the natural emittance of a lattice consisting entirely of FODO cells:

$$\varepsilon_0 \approx C_q \gamma^2 \left(\frac{2f}{L} \right)^3 \theta^3$$

Notice how the emittance scales with the beam and lattice parameters:

- The emittance is proportional to the *square* of the energy.
- The emittance is proportional to the *cube* of the bending angle. Increasing the number of cells in a complete circular lattice reduces the bending angle of each dipole, and reduces the emittance.
- The emittance is proportional to the *cube* of the quadrupole focal length. Stronger quadrupoles have shorter focal lengths, and reduce the emittance.
- The emittance is inversely proportional to the *cube* of the cell (or dipole) length. Shortening the cell reduces the lattice functions, and reduces the emittance.

FODO lattice - natural emittance

Recall that the phase advance in a FODO cell is given by:

$$\cos\mu_x = 1 - \frac{L^2}{2f^2}$$

This means that a stable lattice must have: $\frac{f}{L} \geq \frac{1}{2}$

In the limiting case, $\mu_x = 180$, and we have the minimum value for f: $f = \frac{L}{2}$

Using our approximation:

$$\varepsilon_0 \approx C_q \gamma^2 \left(\frac{2f}{L} \right)^3 \theta^3$$

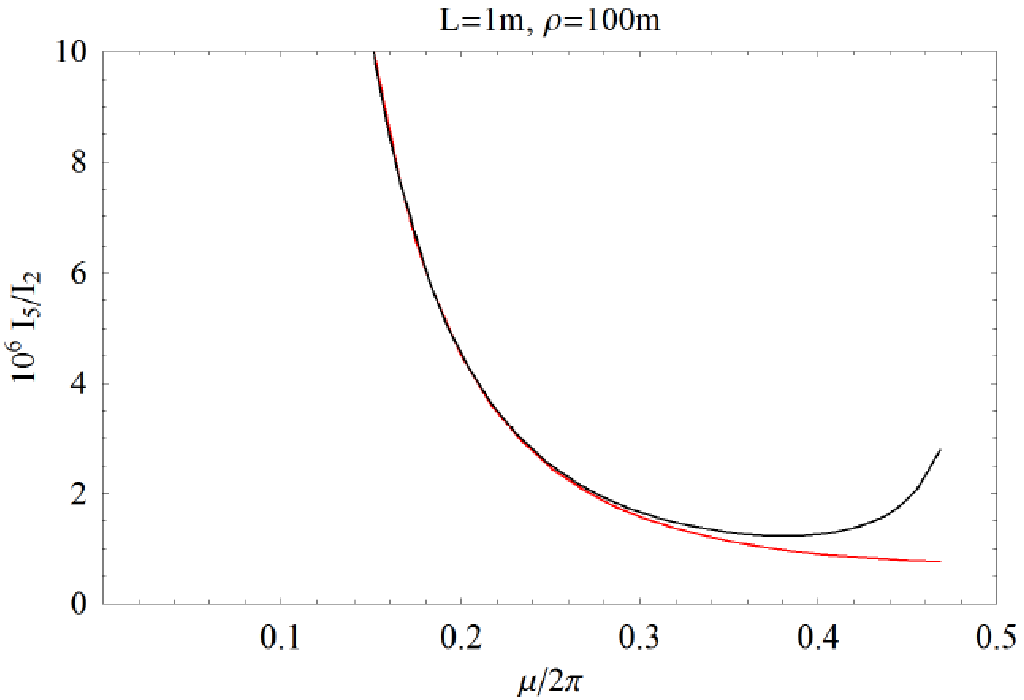
this would suggest that the *minimum emittance in a FODO lattice* is given by:

$$\varepsilon_0 \approx C_q \gamma^2 \theta^3$$

However, as we increase the focusing strength, the approximations we used to obtain this simple form for ε_0 break down...

FODO lattice - natural emittance

Plotting the exact formula for I_5/I_2, as a function of the phase advance, we find there is a minimum in the natural emittance, for $\mu \approx 137°$.



Black line: exact formula

Red line: approximation,

$$\frac{I_5}{I_2} \approx \left(1 - \frac{L^2}{16 f^2}\right) \frac{8 f^3}{\rho^3}$$

It turns out that the minimum value the natural emittance in a FODO cell is given by:

$$\varepsilon_0 \approx 1.2 C_q \gamma^2 \theta^3$$

FODO lattice - natural emittance

A phase advance of 137° is quite high for a FODO cell. More typically, beam lines are designed with a phase advance of 90° per cell.

For a 90° FODO cell:

$$\cos\mu_x = 1 - \frac{L^2}{2f^2} = 0 \qquad \therefore \qquad \frac{f}{L} = \frac{1}{\sqrt{2}}$$

We are just in the regime where our approximation $4f >> L$ is valid; so in this case:

$$\varepsilon_0 \approx C_q \gamma^2 \left(\frac{2f}{L}\right)^3 \theta^3 = 2\sqrt{2} C_q \gamma^2 \theta^3$$

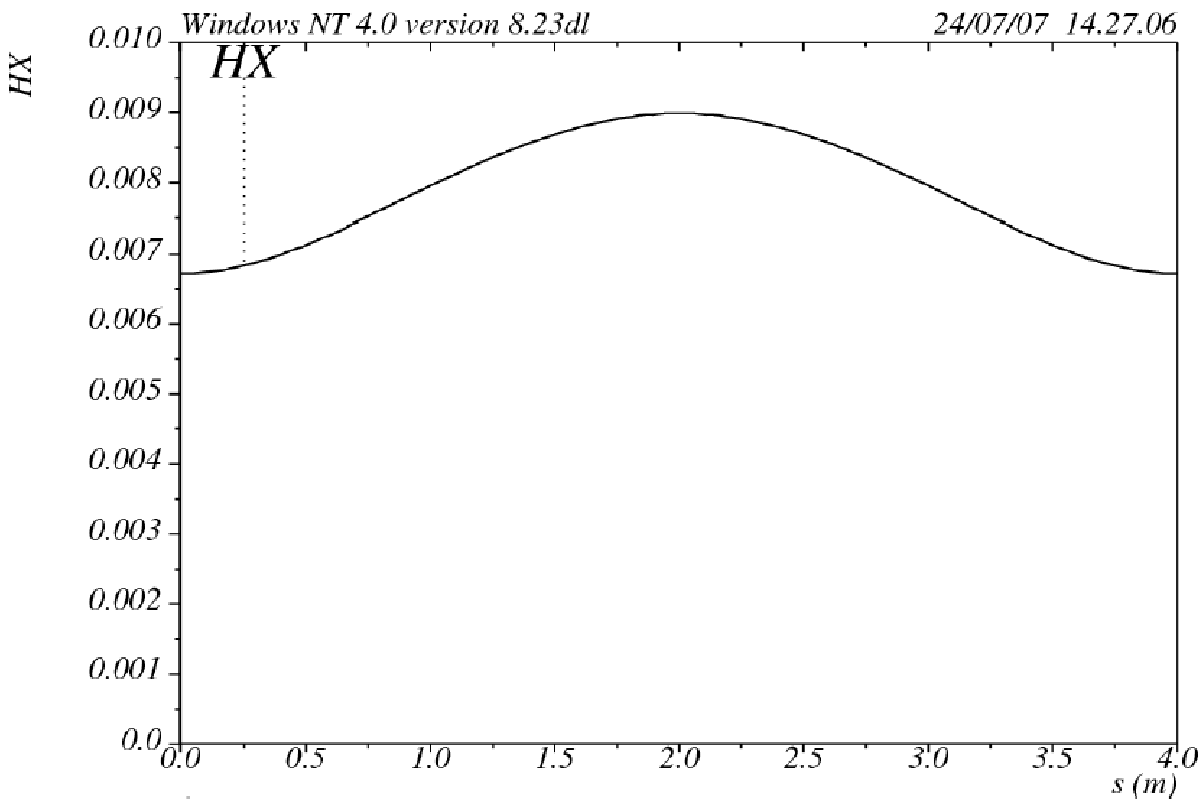
Using the above formulae, we estimate that a storage ring constructed from 16 FODO cells with 90 phase advance per cell, and storing beam at 2 GeV would have a natural emittance of 125 nm.

Many modern applications (including light sources and colliders) demand emittances one or two orders of magnitude smaller.

How can we design the lattice to achieve a smaller natural emittance?
A clue is provided if we look at the curly-H function in a FODO lattice…

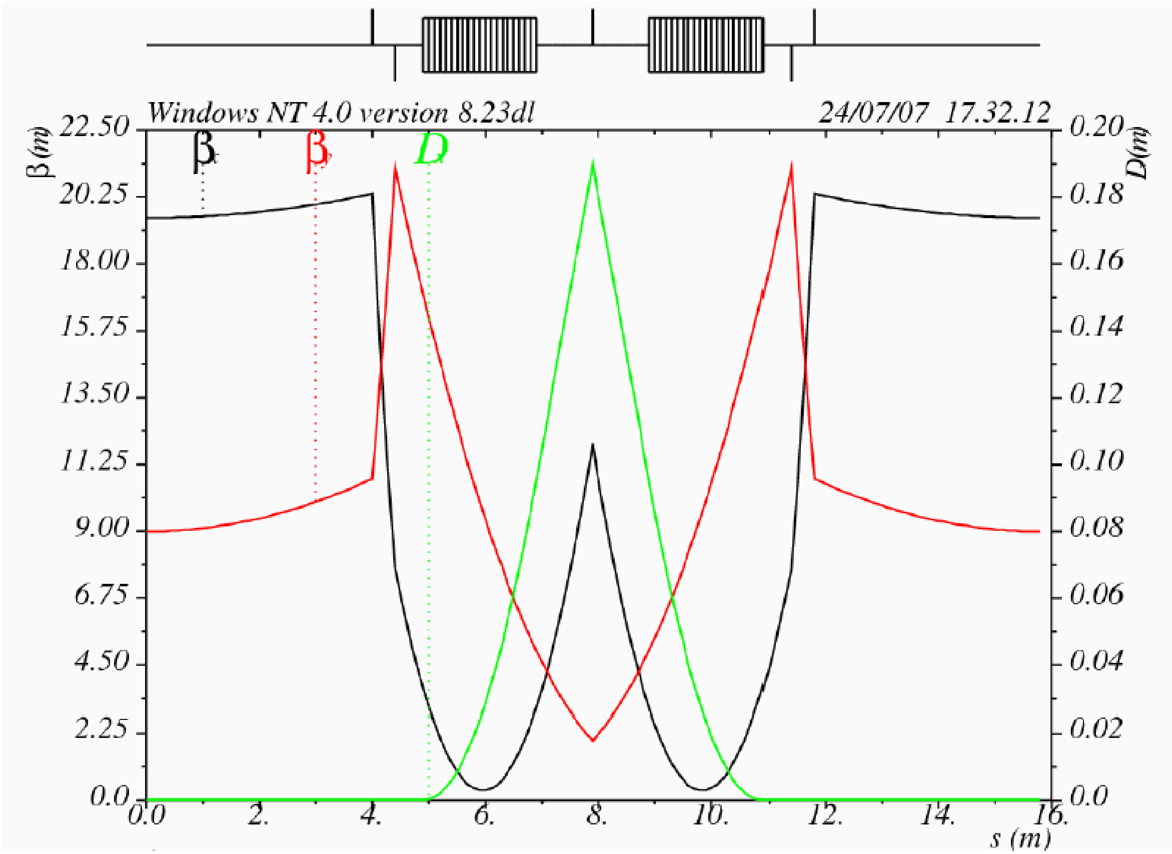
FODO lattice - natural emittance

The curly-H function remains at a relatively constant value throughout the lattice. Perhaps we can reduce it in the dipoles…



DBA lattice - natural emittance

As a first attempt at reducing the natural emittance, let us try designing a lattice that has zero dispersion at one end of each dipole. This can be achieved using a double bend achromat (DBA) lattice.



DBA lattice - natural emittance

First of all, let us consider the constraints needed to achieve zero dispersion at either end of the cell.

Assuming that we start at one end of the cell with zero dispersion, then, by symmetry, the dispersion at the other end of the cell will also be zero if the central quadrupole simply reverses the gradient of the dispersion.

In the thin lens approximation, this condition can be written:

$$\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \cdot \begin{pmatrix} \eta_x \\ \eta_{px} \end{pmatrix} = \begin{pmatrix} \eta_x \\ \eta_{px} - \dfrac{\eta_x}{f} \end{pmatrix} = \begin{pmatrix} \eta_x \\ -\eta_{px} \end{pmatrix}$$

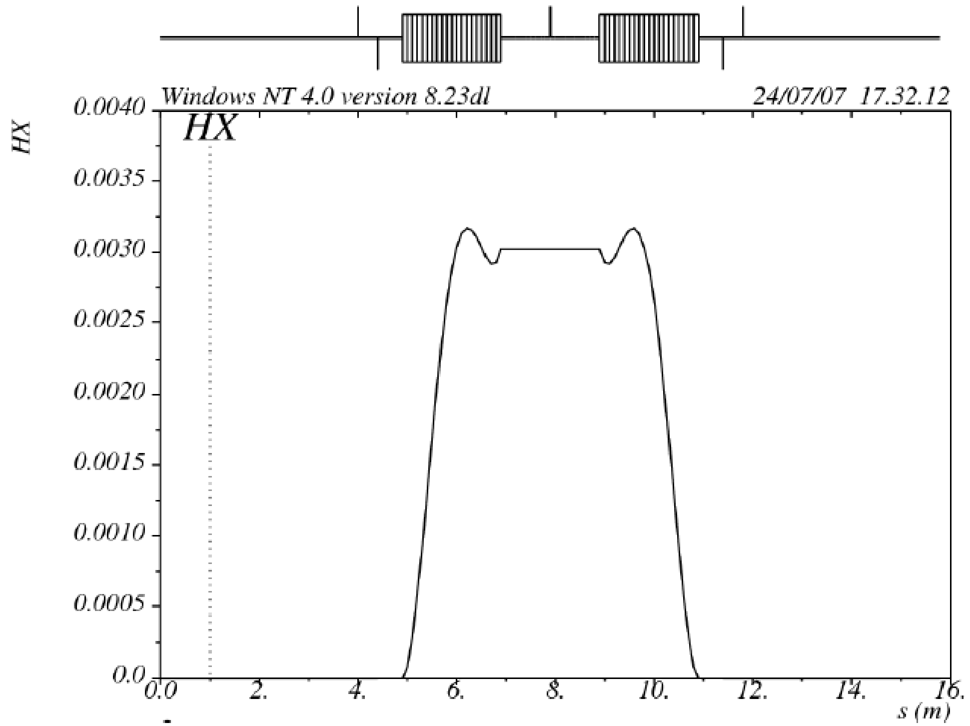
Hence, the central quadrupole must have focal length: $f = \dfrac{\eta_x}{2\eta_{px}}$

The actual value of the dispersion is determined by the dipole bending angle θ, the bending radius ρ, and the drift length L:

$$\eta_x = \rho(1-\cos\theta) + L\sin\theta \qquad \eta_{px} = \sin\theta$$

DBA lattice - natural emittance

Is this type of lattice likely to have a lower natural emittance than a FODO lattice? We can get an idea by looking at the curly-H function.



Note that we use the same dipoles (bending radius and length) for our example in both cases (FODO and DBA). In the DBA lattice the curly-H function is reduced by a significant factor, compared to the FODO lattice.

DBA lattice - natural emittance

Let us calculate the minimum natural emittance of a DBA lattice, for given bending radius ρ and bending angle θ in the dipoles.

To do this, we need to calculate the minimum value of:

$$I_5 = \int \frac{\mathcal{H}_x}{\rho^3} ds$$

in one dipole, subject to the constraints:

$$\eta_0 = \eta_{p0} = 0$$

where η_0 and η_{p0} are the dispersion and the gradient of the dispersion at the entrance of a dipole.

We know how the dispersion and the Twiss parameters evolve through the dipole, so we can calculate I_5 for one dipole, for given initial values of the Twiss parameters α_0 and β_0.

Then, we simply have to minimise the value of I_5 with respect to α_0 and β_0.

Again, the algebra is rather formidable, and the full expression for I_5 is not especially enlightening…

DBA lattice - natural emittance

We find that, for given ρ and θ and with the constraints:

$$\eta_0 = \eta_{p0} = 0$$

the minimum value of I_5 is given by:

$$I_{5,\min} = \frac{1}{4\sqrt{15}} \frac{\theta^4}{\rho} + O(\theta^6)$$

which occurs for values of the Twiss parameters at the entrance to the dipole:

$$\beta_0 = \sqrt{\frac{12}{5}} L + O(\theta^3) \qquad \alpha_0 = \sqrt{15} + O(\theta^2)$$

where $L = \rho\theta$ is the length of a dipole.

Since:

$$I_2 = \int \frac{1}{\rho^2} ds = \frac{\theta}{\rho}$$

we can immediately write an expression for the minimum emittance in a DBA lattice…

DBA lattice - natural emittance

$$\varepsilon_{0,DBA,\min} = C_q \gamma^2 \frac{I_{5,\min}}{j_x I_2} \approx \frac{1}{4\sqrt{15}} C_q \gamma^2 \theta^3$$

The approximation is valid for small θ. Note that we have again assumed that, since there is no quadrupole component in the dipole, $j_x \approx 1$.

Compare the above expression with that for the minimum emittance in a FODO lattice:

$$\varepsilon_{0,FODO,\min} \approx C_q \gamma^2 \theta^3$$

The minimum emittance in each case scales with the square of the beam energy, and with the cube of the bending angle of a dipole. However, the minimum emittance in a DBA lattice is smaller than that in a FODO lattice (for given energy and dipole bending angle) by a factor $4\sqrt{15} \approx 15.5$.

This is a significant improvement… but can we do even better?

TME lattice - natural emittance

We used the constraints:

$$\eta_0 = \eta_{p0} = 0$$

to define a DBA lattice; but to get a lower emittance, we can consider relaxing these constraints.

If we relax these constraints, then we may be able to achieve an even lower natural emittance.

To derive the "theoretical minimum emittance" (TME), we write down an expression for:

$$I_5 = \int \frac{\mathcal{H}_x}{\rho^3} ds$$

with arbitrary initial dispersion η_0, η_{p0}, and Twiss parameters α_0 and β_0 in a dipole with given bending radius ρ and angle θ.

Then we minimise I_5 with respect to variations in η_0, η_{p0}, α_0 and β_0…

TME lattice - natural emittance

The result is:

$$\varepsilon_{0,TME,\min} \approx \frac{1}{12\sqrt{15}} C_q \gamma^2 \theta^3$$

The minimum emittance is obtained with dispersion at the entrance to a dipole:

$$\eta_0 = \frac{1}{6} L\theta + O(\theta^3) \qquad \eta_{p0} = -\frac{\theta}{2} + O(\theta^3)$$

and with Twiss functions at the entrance:

$$\beta_0 = \frac{8}{\sqrt{15}} L + O(\theta^3) \qquad \alpha_0 = \sqrt{15} + O(\theta^2)$$

TME lattice - natural emittance

Note that with the conditions for minimum emittance:

$$\eta_0 = \frac{1}{6} L\theta + O(\theta^3) \qquad \eta_{p0} = -\frac{\theta}{2} + O(\theta^3)$$

$$\beta_0 = \frac{8}{\sqrt{15}} L + O(\theta^3) \qquad \alpha_0 = \sqrt{15} + O(\theta^2)$$

the dispersion and the beta function reach a minimum in the centre of the dipole. The values at the centre of the dipole are:

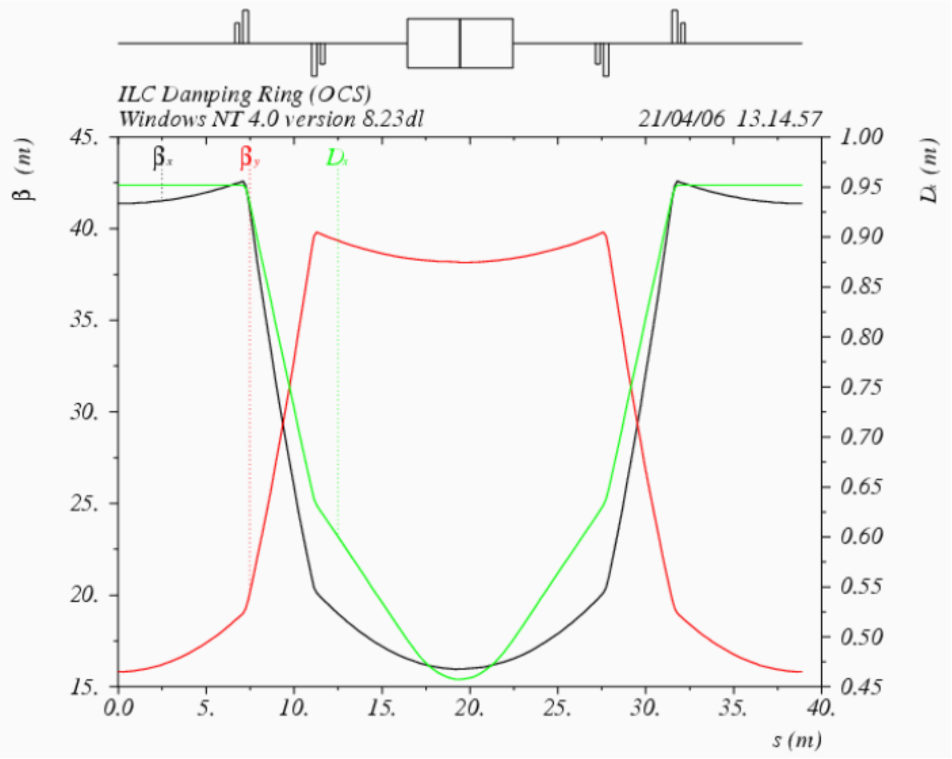
$$\eta_{\min} = \rho\left(1 - 2\frac{\sin\frac{\theta}{2}}{\theta}\right) = \frac{L\theta}{24} + O(\theta^4)$$

$$\beta_{\min} = \frac{L}{2\sqrt{15}} + O(\theta^3)$$

What do the lattice functions look like in a single cell of a TME lattice?

Because of symmetry in the dipole, we can consider a TME lattice cell as containing a single dipole (as opposed to two dipoles, which we had in the cases of the FODO and DBA lattices)…

TME lattice - natural emittance



Note: the lattice shown in this example does not actually achieve the exact conditions needed for absolute minimum emittance. A more complicated lattice would be needed for this…

Summary: FODO, DBA and TME lattices

Lattice Style	Minimum Emittance	Conditions
90 FODO	$\varepsilon_0 \approx 2\sqrt{2} C_q \gamma^2 \theta^3$	$\frac{f}{L} = \frac{1}{\sqrt{2}}$
Minimum emittance FODO	$\varepsilon_0 \approx 1.2 C_q \gamma^2 \theta^3$	$\mu \approx 137^\circ$
DBA	$\varepsilon_0 \approx \frac{1}{4\sqrt{15}} C_q \gamma^2 \theta^3$	$\eta_0 = \eta_{p0} = 0$ $\beta_0 \approx \sqrt{12/5} L$ $\alpha_0 \approx \sqrt{15}$
TME	$\varepsilon_0 \approx \frac{1}{12\sqrt{15}} C_q \gamma^2 \theta^3$	$\eta_{\min} \approx \frac{L\theta}{24}$ $\beta_{\min} \approx \frac{L}{2\sqrt{15}}$

Note: the approximations are valid for small dipole bending angle, θ.

Design for low emittance lattices

The results we have derived have been for "ideal" lattices that perfectly achieve the stated conditions in each case.

In practice, lattices rarely, if ever, achieve the ideal conditions. In particular, the beta function in an achromat is usually not optimal for low emittance; and the dispersion and beta function in a TME lattice are not optimal.

The main reasons for this are:

- It is difficult to control the beta function and dispersion to achieve the ideal low-emittance conditions with a small number of quadrupoles.
- There are other strong dynamical constraints on the design that we have not considered: in particular, the lattice needs a large dynamic aperture to achieve a good beam lifetime.

The dynamic aperture issue is particularly difficult for low emittance lattices. The dispersion in low emittance lattices is generally low, while the strong focusing leads to high chromaticity. Therefore, very strong sextupoles are often needed to correct the natural chromaticity. This limits the dynamic aperture.

The consequence of all these issues is that in practice, the natural emittance of a lattice of a given type is usually somewhat larger than might be expected using the formulae given here.

Summary

The natural emittance in a storage ring is determined by the balance between the radiation damping (given by I_2) and the quantum excitation (given by I_5).

The quantum excitation depends on the lattice functions. Different "styles" of lattice can be used, depending on the emittance specification for the storage ring.

In general, for small bending angle θ the natural emittance can be written as:

$$\varepsilon_0 \approx F C_q \gamma^2 \theta^3$$

where θ is the bending angle of a single dipole, and the numerical factor F is determined by the lattice style:

Lattice style	F
90 FODO	$2\sqrt{2}$
180 FODO	1
Double-bend achromat (DBA)	$1/4\sqrt{15}$
Multi-bend achromat	$(M+1)/12\sqrt{15}(M-1)$
Theoretical minimum emittance (TME)	$1/12\sqrt{15}$

Summary cont.

Achromats have been popular choices for storage ring lattices in third-generation synchrotron light sources for two reasons:

- they provide lower natural emittance than FODO lattices;
- they provide zero-dispersion locations appropriate for insertion devices (wigglers and undulators).

Light sources using double-bend achromats (e.g. ESRF, APS, SPring-8, DIAMOND, SOLEIL…) and triple-bend achromats (e.g. ALS, SLS) have been built.

Increasing the number of bends in a single cell of an achromat ("multiple-bend achromats") reduces the emittance, since the lattice functions in the "central" bends can be tuned to conditions for minimum emittance.

"Detuning" an achromat to allow some dispersion in the straights provides the possibility of further reduction in natural emittance, by moving towards the conditions for a theoretical minimum emittance (TME) lattice.