Accelerator Physics

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Lecture 14

Statistical Treatments of Beams

- Distribution Functions Defined
  - Statistical Averaging
  - Examples
- Kinetic Equations
  - Liouville Theorem
  - Vlasov Theory
  - Collision Corrections
- Self-consistent Fields
- Collective Effects
  - KV Equation
  - Landau Damping
- Beam-Beam Effect

Beam *rms* Emittance

Treat the beam as a statistical ensemble as in Statistical Mechanics. Define the distribution of particles within the beam statistically. Define single particle distribution function

$$\psi\left(x,x'\right),$$

where $\psi(x,x')dxdx'$ is the number of particles in $[x,x+dx]$ and $[x',x'+dx']$, and statistical averaging as in Statistical Mechanics, e. g.

$$\left\langle q\right\rangle \equiv \int q\left(x,x'\right)\psi\left(x,x'\right)dxdx'/N$$
$$\left\langle q^2\right\rangle \equiv \int q^2\left(x,x'\right)\psi\left(x,x'\right)dxdx'/N$$
$$\vdots$$

Closest *rms* Fit Ellipses

For zero-centered distributions, i.e., distributions that have zero average value for x and x'

$$\varepsilon_{rms} \equiv \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$

$$\beta = \frac{\langle x^2 \rangle}{\varepsilon_{rms}} = \frac{\sigma_x^2}{\varepsilon_{rms}}$$

$$\alpha = -\frac{\langle xx' \rangle}{\varepsilon_{rms}}$$

$$\gamma = \frac{\langle x'^2 \rangle}{\varepsilon_{rms}} = \frac{\sigma_{x'}^2}{\varepsilon_{rms}}$$

Case: Uniformly Filled Ellipse

$$\psi\left(x,x'\right)=\frac{1}{\pi\varepsilon}\Theta\left(1-\frac{\gamma x^2+2\alpha x x'+\beta x'^2}{\varepsilon}\right)$$

Θ here is the Heavyside step function, 1 for positive values of its argument and zero for negative values of its argument

$$\sigma_x^2=\left\langle x^2\right\rangle=\frac{\varepsilon\beta}{4}$$

$$\left\langle x x'\right\rangle=-\frac{\varepsilon\alpha}{4}$$

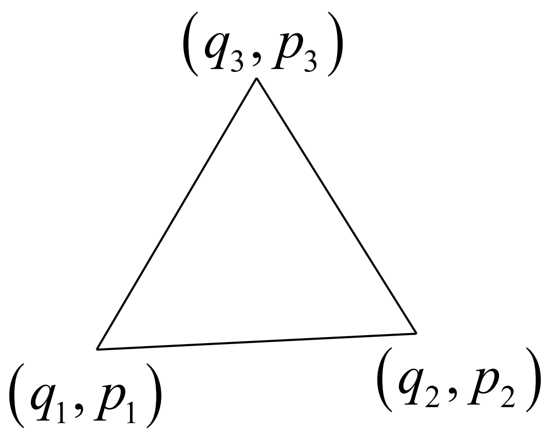
$$\sigma_{x'}^2=\left\langle x'^2\right\rangle=\frac{\varepsilon}{4\beta}\left(1+\alpha^2\right)$$

$$\therefore\ \varepsilon_{rms}=\frac{\varepsilon}{4}$$

Gaussian models (HW) are good, especially for lepton machines

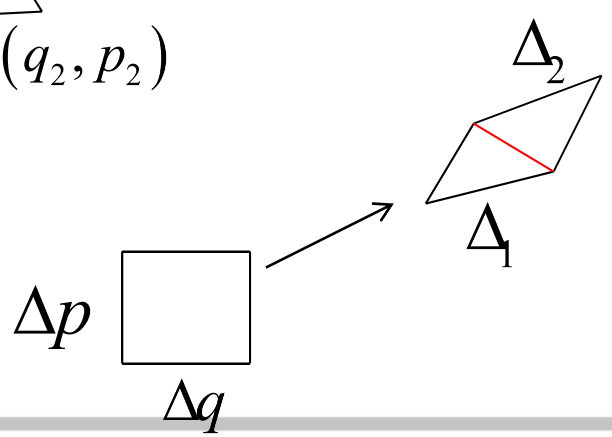
Dynamics? Start with Liouville Thm.

Generalization of the Area Theorem of Linear Optics. Simple Statement: For a dynamical system that may be described by a conserved energy function (Hamiltonian), the relevant phase space volume is conserved by the flow, *even if the forces are non-linear*. Start with some simple geometry!



$$\text{Area } \Delta = \frac{(q_2 - q_1)(p_3 - p_1)}{2} - \frac{(q_3 - q_1)(p_2 - p_1)}{2}$$

$$(\text{acute angle has line } 1-2 \text{ clockwise wrt line } 1-3)$$



In phase space
Area Before=Area After

Liouville Theorem

$$(q_0, p_0) \rightarrow \left(q_0 + \frac{\partial H}{\partial p}(q_0, p_0)\Delta t + \cdots, p_0 - \frac{\partial H}{\partial q}(q_0, p_0)\Delta t + \cdots \right)$$

$$(q_0 + \Delta q, p_0) \rightarrow \left(q_0 + \Delta q + \frac{\partial H}{\partial p}(q_0 + \Delta q, p_0)\Delta t + \cdots, p_0 - \frac{\partial H}{\partial q}(q_0 + \Delta q, p_0)\Delta t + \cdots \right)$$

$$(q_0, p_0 + \Delta p) \rightarrow \left(q_0 + \frac{\partial H}{\partial p}(q_0, p_0 + \Delta p)\Delta t + \cdots, p_0 + \Delta p - \frac{\partial H}{\partial q}(q_0, p_0 + \Delta p)\Delta t + \cdots \right)$$

$$(q_0 + \Delta q, p_0 + \Delta p) \rightarrow \begin{pmatrix} q_0 + \Delta q + \dfrac{\partial H}{\partial p}(q_0 + \Delta q, p_0 + \Delta p)\Delta t + \cdots, \\ p_0 + \Delta p - \dfrac{\partial H}{\partial q}(q_0 + \Delta q, p_0 + \Delta p)\Delta t + \cdots \end{pmatrix}$$

$$\therefore \text{Area } \Delta_1 = \frac{1}{2}\det \begin{vmatrix} \Delta q + \left[\dfrac{\partial H}{\partial p}(q_0 + \Delta q, p_0) - \dfrac{\partial H}{\partial p}(q_0, p_0) \right]\Delta t & -\left[\dfrac{\partial H}{\partial q}(q_0 + \Delta q, p_0) - \dfrac{\partial H}{\partial q}(q_0, p_0) \right]\Delta t \\ \left[\dfrac{\partial H}{\partial p}(q_0, p_0 + \Delta p) - \dfrac{\partial H}{\partial p}(q_0, p_0) \right]\Delta t & \Delta p - \left[\dfrac{\partial H}{\partial q}(q_0, p_0 + \Delta p) - \dfrac{\partial H}{\partial q}(q_0, p_0) \right]\Delta t \end{vmatrix}$$

$$\underset{\Delta t \to 0}{\rightarrow} \frac{1}{2}(\Delta q \Delta p)\left[1 + \left\{ \frac{\partial^2 H}{\partial p \partial q}(q_0, p_0) - \frac{\partial^2 H}{\partial p \partial q}(q_0, p_0) \right\}\Delta t \right] = \frac{(\Delta q \Delta p)}{2}$$

Likewise

$$\text{Area } \Delta_2 = \frac{1}{2}\det\begin{vmatrix} \Delta q + \left[\frac{\partial H}{\partial p}(q_0+\Delta q, p_0) - \frac{\partial H}{\partial p}(q_0+\Delta q, p_0+\Delta p)\right]\Delta t & -\left[\frac{\partial H}{\partial q}(q_0+\Delta q, p_0) - \frac{\partial H}{\partial q}(q_0+\Delta q, p_0+\Delta p)\right]\Delta t \\ \left[\frac{\partial H}{\partial p}(q_0, p_0+\Delta p) - \frac{\partial H}{\partial p}(q_0+\Delta q, p_0+\Delta p)\right]\Delta t & \Delta p - \left[\frac{\partial H}{\partial q}(q_0, p_0+\Delta p) - \frac{\partial H}{\partial q}(q_0+\Delta q, p_0+\Delta p)\right]\Delta t \end{vmatrix}$$

$$\underset{\Delta t \to 0}{\to} \frac{1}{2}(\Delta q \Delta p)\left[1 + \left\{\frac{\partial^2 H}{\partial p \partial q}(q_0+\Delta p, p_0+\Delta p) - \frac{\partial^2 H}{\partial p \partial q}(q_0+\Delta p, p_0+\Delta p)\right\}\Delta t\right] = \frac{(\Delta q \Delta p)}{2}$$

Because the starting point is arbitrary, phase space area is conserved at each location in phase space. In three dimensions, the full 6-D phase volume is conserved by essentially the same argument, as is the sum of the projected areas in each individual projected phase spaces (the so-called third Poincare and first Poincare invariants, respectively). Defeat it by adding non-Hamiltonian (dissipative!) terms later.

Vlasov Equation

By interpretation of ψ as the single particle distribution function, and because the individual particles in the distribution are assumed to not cross the boundaries of the phase space volumes (collisions neglected!), ψ must evolve so that

$$\frac{d\psi}{dt} = 0 \qquad \text{as the distibution evolves}$$

$$\frac{d\psi}{dt} = \lim_{\delta t \to 0} \frac{\psi\left(t+\delta t, \vec{q}\left(t+\delta t\right), \vec{p}\left(t+\delta t\right)\right) - \psi\left(t, \vec{q}\left(t\right), \vec{p}\left(t\right)\right)}{\delta t} = 0$$

where the equation for ANY (this is what makes it hard to solve in general!) individual orbits through phase space is given by $\vec{q}\left(t\right), \vec{p}\left(t\right)$

$$\therefore \frac{\partial \psi}{\partial t} + \frac{d\vec{q}}{dt}\frac{\partial \psi}{\partial \vec{q}} + \frac{d\vec{p}}{dt}\frac{\partial \psi}{\partial \vec{p}} = 0$$

Conservation of Probability

$N(t)=\int \psi(t;\vec{q},\vec{p})d^3\vec{x}d^3\vec{p}$ is a conserved quantity

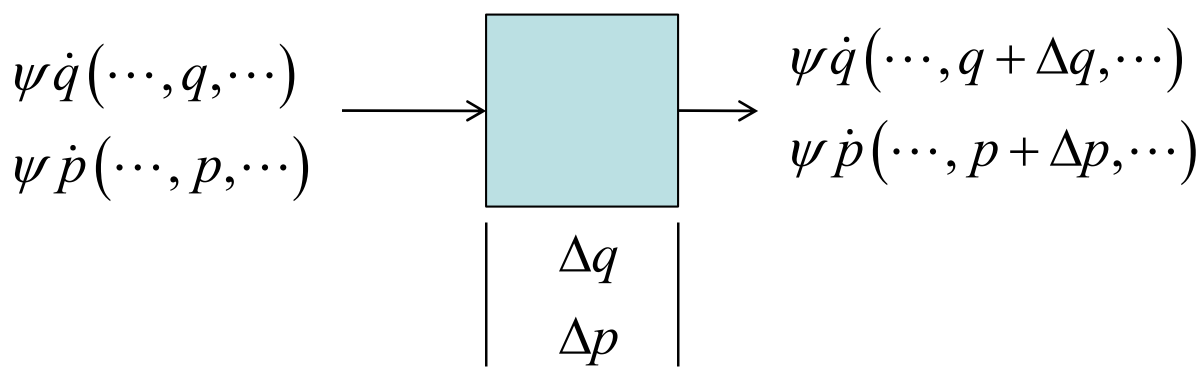
continuity equation for ψ is

$$\frac{\partial\psi}{\partial t}+\nabla_{\vec{q}}\left(\dot{\vec{q}}\psi\right)+\nabla_{\vec{p}}\left(\dot{\vec{p}}\psi\right)=0$$

$$\therefore \frac{\partial\psi}{\partial t}+\frac{d\vec{q}}{dt}\frac{\partial\psi}{\partial\vec{q}}+\frac{d\vec{p}}{dt}\frac{\partial\psi}{\partial\vec{p}}+\psi\left[\nabla_{\vec{q}}\frac{\partial H}{\partial\vec{p}}-\nabla_{\vec{p}}\frac{\partial H}{\partial\vec{q}}\right]=0$$

for the Hamiltonian system

$$\therefore \frac{\partial\psi}{\partial t}+\frac{d\vec{q}}{dt}\frac{\partial\psi}{\partial\vec{q}}+\frac{d\vec{p}}{dt}\frac{\partial\psi}{\partial\vec{p}}=0$$



Jean's Theorem

The independent variable in the Vlasov equation is often changed to the variable s. In this case the Vlasov equation is

$$\frac{\partial \psi}{\partial s}+\frac{d\vec{q}}{ds}\frac{\partial \psi}{\partial \vec{q}}+\frac{d\vec{p}}{ds}\frac{\partial \psi}{\partial \vec{p}}=0$$

The equilibrium Vlasov problem, $\partial\psi/\partial t=0$, is solved by any function of the constants of the motion. This result is called Jean's theorem, and is the starting point for instability analysis as the "unperturbed problem".

If $\psi = f\left(A,B,C,\cdots\right)$, where $A,B,C,\cdots$ are constants of the motion

$$\frac{d\vec{x}}{dt}\frac{\partial \psi}{\partial \vec{x}}+\frac{d\vec{p}}{dt}\frac{\partial \psi}{\partial \vec{p}}=\frac{\partial f}{\partial A}\left(\frac{d\vec{x}}{dt}\frac{\partial A}{\partial \vec{x}}+\frac{d\vec{p}}{dt}\frac{\partial A}{\partial \vec{p}}\right)+\frac{\partial f}{\partial B}\left(\frac{d\vec{x}}{dt}\frac{\partial B}{\partial \vec{x}}+\frac{d\vec{p}}{dt}\frac{\partial B}{\partial \vec{p}}\right)$$

$$+\frac{\partial f}{\partial C}\left(\frac{d\vec{x}}{dt}\frac{\partial C}{\partial \vec{x}}+\frac{d\vec{p}}{dt}\frac{\partial C}{\partial \vec{p}}\right)+\cdots=\frac{\partial f}{\partial A}\frac{dA}{dt}+\frac{\partial f}{\partial B}\frac{dB}{dt}+\frac{\partial f}{\partial C}\frac{dC}{dt}+\cdots=0$$

Examples

1-D Harmonic oscillator Hamiltonian. Bi-Maxwellian distribution is a stationary distribution

$$\psi = \frac{1}{2\pi}\exp\left(-H / kT\right) = \frac{m\omega}{2\pi kT}\exp\left(-p_x^2 / 2mkT\right)\exp\left(-mx^2\omega^2 / 2kT\right),$$

As is any other function of the Hamiltonian. Contours of constant ψ line up with contours of constant H

2 D transverse Gaussians, including focusing structure in ring

$$\psi\left(s;x,x';y,y'\right) \propto \exp\left(-\left(\gamma_x\left(s\right)x^2 + 2\alpha_x\left(s\right)xx' + \beta_x\left(s\right)x'^2\right)/\varepsilon_x\right)$$
$$\times\exp\left(-\left(\gamma_y\left(s\right)y^2 + 2\alpha_y\left(s\right)yy' + \beta_y\left(s\right)y'^2\right)/\varepsilon_y\right)$$

Contours of constant ψ line up with contours of constant Courant-Snyder invariant. Stationary as particles move on ellipses!

Solution by Characteristics

More subtle: a solution to the full Vlasov equation may be obtained from the distribution function at some the initial condition, provided the particle orbits may be found unambiguously from the initial conditions throughout phase space. Example: 1-D harmonic oscillator Hamiltonian.

$$\begin{pmatrix} x(t) \\ x'(t) \end{pmatrix} = \begin{pmatrix} \cos\omega(t-t_0) & \sin\omega(t-t_0)/\omega \\ -\omega\sin\omega(t-t_0) & \cos\omega(t-t_0) \end{pmatrix}\begin{pmatrix} x(t_0) \\ x'(t_0) \end{pmatrix} \to \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \begin{pmatrix} \cos\omega(t-t_0) & -\sin\omega(t-t_0)/\omega \\ \omega\sin\omega(t-t_0) & \cos\omega(t-t_0) \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix}$$

$$\psi(x,x';t=t_0) = f_0(x,x')$$

Let $\psi(x,x';t) = f_0\left(\cos\omega(t-t_0)x - \sin\omega(t-t_0)x'/\omega, \omega\sin\omega(t-t_0)x + \cos\omega(t-t_0)x'\right)$

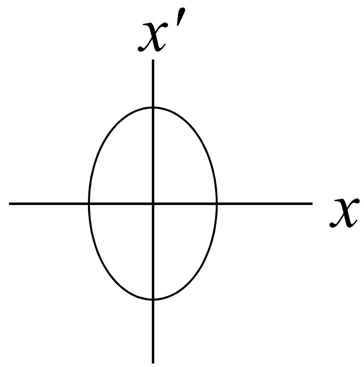
$$\frac{\partial\psi}{\partial t} = \frac{\partial f_0}{\partial x}\frac{dx(t;x,x')}{dt} + \frac{\partial f_0}{\partial x'}\frac{dx'(t;x,x')}{dt}$$

$$= \frac{\partial f}{\partial x}\left[-\omega\sin\omega(t-t_0)x - \cos\omega(t-t_0)x'\right] + \frac{\partial f}{\partial x'}\left[\omega^2\cos\omega(t-t_0)x - \omega\sin\omega(t-t_0)x'\right]$$

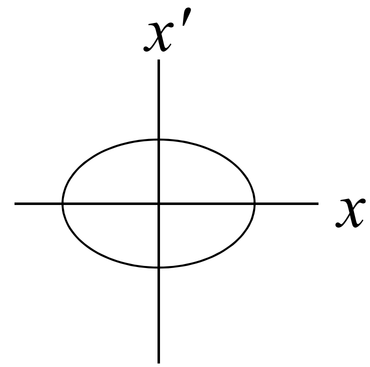
$$\frac{dx}{dt}\frac{\partial\psi}{\partial x} = x'\left[\frac{\partial f}{\partial x}\cos\omega(t-t_0) + \frac{\partial f}{\partial x'}\omega\sin\omega(t-t_0)\right]$$

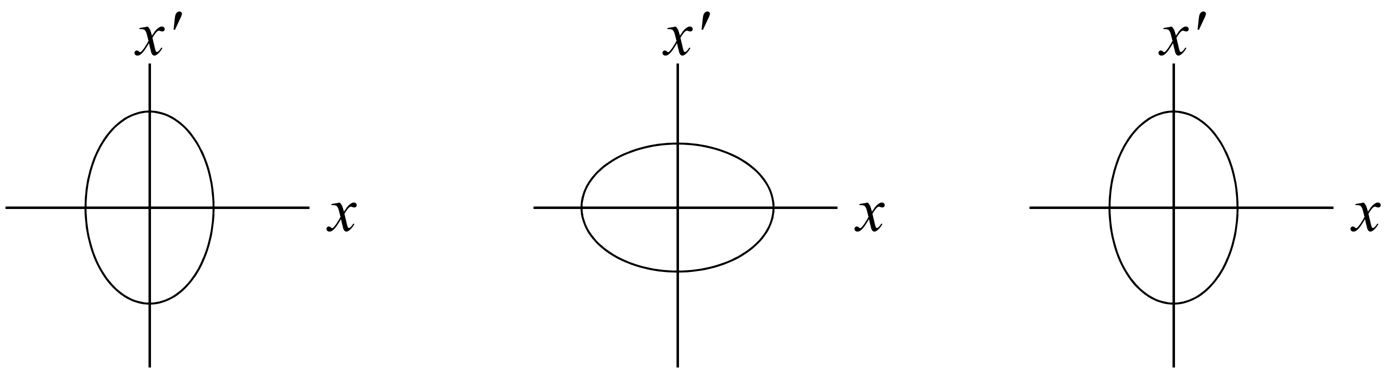
$$\frac{dx'}{dt}\frac{\partial\psi}{\partial x'} = -\omega^2 x\left[-\frac{\partial f}{\partial x}\sin\omega(t-t_0)/\omega + \frac{\partial f}{\partial x'}\cos\omega(t-t_0)\right] \therefore \frac{d\psi}{dt} = 0$$

Breathing Mode



Quarter Oscillation





The particle envelope "breaths" at twice the revolution frequency!

Sacherer Theory

Assume beam is acted on by a linear focusing force plus additional linear or non-linear forces

$$x'' + k_x^2 x - F_x = 0$$

$$y'' + k_y^2 x - F_y = 0$$

For space charge example we'll see

$$F_{x(y)} = \frac{qE_{x(y)}\left(1-\beta^2\right)}{\gamma mc^2\beta^2} = \frac{qE_{x(y)}}{\gamma^3 mc^2\beta^2}$$

Now

$$\left\langle xx'' \right\rangle + k_x^2 \left\langle x^2 \right\rangle - \left\langle F_x x \right\rangle = 0$$

$$\left\langle yy'' \right\rangle + k_y^2 \left\langle y^2 \right\rangle - \left\langle F_y y \right\rangle = 0$$

Assume distributions zero-centered and let

$$\tilde{x}^2 = \left\langle x^2 \right\rangle \quad \tilde{x}'^2 = \left\langle x'^2 \right\rangle \qquad \tilde{y}^2 = \left\langle y^2 \right\rangle \quad \tilde{y}'^2 = \left\langle y'^2 \right\rangle$$

$$\left\langle x^2 \right\rangle' = 2\left\langle xx' \right\rangle = \tilde{x}^{2\prime} = 2\tilde{x}\tilde{x}'$$

$$\left\langle x^2 \right\rangle'' = \tilde{x}^{2\prime\prime} = \left(2\tilde{x}\tilde{x}'\right)' = 2\left(\tilde{x}\tilde{x}'' + \tilde{x}'^2\right)$$

Also

$$\left\langle xx' \right\rangle' = \left\langle x'^2 \right\rangle + \left\langle xx'' \right\rangle = \left\langle x'^2 \right\rangle - k_x^2 \left\langle x^2 \right\rangle + \left\langle F_x x \right\rangle$$

$$\frac{1}{2}\left\langle x^2 \right\rangle'' = \left\langle xx' \right\rangle' = \tilde{x}\tilde{x}'' + \tilde{x}'^2 = \left\langle x'^2 \right\rangle - k_x^2 \left\langle x^2 \right\rangle + \left\langle F_x x \right\rangle$$

$$\tilde{x}' = \left\langle xx' \right\rangle / \tilde{x} \rightarrow \tilde{x}\tilde{x}'' + \frac{\left\langle xx' \right\rangle^2 - \left\langle x'^2 \right\rangle \left\langle x^2 \right\rangle}{\tilde{x}^2} + k_x^2 \tilde{x}^2 - \left\langle F_x x \right\rangle = 0$$

$$\tilde{x}'' + \frac{\left\langle xx' \right\rangle^2 - \left\langle x'^2 \right\rangle \left\langle x^2 \right\rangle}{\tilde{x}^3} + k_x^2 \tilde{x} - \frac{\left\langle F_x x \right\rangle}{\tilde{x}} = 0$$

$$\tilde{x}'' - \frac{\varepsilon_{rms}^2}{\tilde{x}^3} + k_x^2 \tilde{x} - \frac{\left\langle F_x x \right\rangle}{\tilde{x}} = 0 \qquad \text{"Envelope" equation}$$

rms Emittance Conserved

$$\left(\left\langle x^2\right\rangle\left\langle x'^2\right\rangle-\left\langle xx'\right\rangle^2\right)'$$

$$=\left\langle x^2\right\rangle'\left\langle x'^2\right\rangle+\left\langle x^2\right\rangle\left\langle x'^2\right\rangle'-2\left\langle xx'\right\rangle\left\langle xx'\right\rangle'$$

$$=2\left\langle xx'\right\rangle\left\langle x'^2\right\rangle+2\left\langle x^2\right\rangle\left\langle x'x''\right\rangle-2\left\langle xx'\right\rangle\left(\left\langle x'^2\right\rangle-k_x^2\left\langle x^2\right\rangle+\left\langle F_x x\right\rangle\right)$$

$$=2\left\langle x^2\right\rangle\left(-k_x^2\left\langle x'x\right\rangle+\left\langle F_x x'\right\rangle\right)+2\left\langle xx'\right\rangle\left(k_x^2\left\langle x^2\right\rangle-\left\langle F_x x\right\rangle\right)$$

$$=2\left\langle x^2\right\rangle\left\langle F_x x'\right\rangle-2\left\langle xx'\right\rangle\left\langle F_x x\right\rangle$$

For linear forces derivative vanishes and *rms* emittance conserved. Emittance growth implies *non-linear forces*.