Accelerator Physics

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Lecture 1

Course Outline

- **Course Content**
- Introduction to Accelerators and Short Historical Overview
  - Basic Units and Definitions
  - Lorentz Force
  - Linear Accelerators
  - Circular Accelerators
- Particle Motion in EM Fields
  - Magnetic Multipoles
  - Linear Beam Dynamics
  - Periodic Systems
  - Nonlinear Perturbations
  - Coupled Motion

- Synchrotron Radiation
    - Radiation Power and Distribution
    - Insertion Devices
    - X-ray Sources
    - Free Electron Lasers
- Technical Components
    - Particle Acceleration Cavities and RF Systems
    - Spin and Spin Manipulation
- Collective Effects
    - Particle Distributions
    - Vlasov Equation
    - Self-consistent Fields

Landau Damping

Beam-Beam Effects

- Relaxation Phenomena

Radiation Damping

Toushek effect/IBS

Beam Cooling

Energy Units

- When a particle is accelerated, i.e., its energy is changed by an electromagnetic field, it must have fallen through an Electric Field (we show later by very general arguments that Magnetic Fields cannot change particle energy). For electrostatic accelerating fields the energy change is

$$\Delta E = q\Delta\Phi = q\left(\Phi_a - \Phi_b\right)$$

q charge, Φ, the electrostatic potentials before and after the motion through the electric field. Therefore, particle energy can be conveniently expressed in units of the "equivalent" electrostatic potential change needed to accelerate the particle to the given energy. Definition: 1 eV, or 1 electron volt, is the energy acquired by 1 electron falling through a one volt potential difference.

Energy Units

$$1\ \text{eV} = 1.6\times10^{-19}\ \text{C} \times 1\ \text{V} = 1.6\times10^{-19}\ \text{J}$$

$$1\ \text{MeV} = 10^{6}\ \text{eV} = 1.6\times10^{-13}\ \text{J}$$

To convert rest mass to eV use Einstein relation

$$E_0 = mc^2$$

where m is the rest mass. For electrons

$$E_{electron,0} = 9.1\times10^{-31}\ \text{kg}\left(3\times10^{8}\ \text{m/sec}\right)^2 = 81.9\times10^{-15}\ \text{J}$$

$$= 0.512\ \text{MeV}$$

Recent "best fit" value 0.51099906 MeV

Some Needed Relativity

Following Maxwell Equations, which exhibit this symmetry, assume all Laws of Physics must be of form to guarantee the invariance of the space-time interval

$$(ct')^2 - x'^2 - y'^2 - z'^2 = (ct)^2 - x^2 - y^2 - z^2$$

Coordinate transformations that leave interval unchanged are the usual rotations and Lorentz Transformations, e.g. the *z* boost

$$ct' = \gamma(ct - \beta z)$$

$$x' = x$$

$$y' = y$$

$$z' = \gamma(z - \beta ct)$$

Relativistic Factors

where, following Einstein define the relativistic factors

$$\vec{\beta} = \frac{\vec{v}}{c} \qquad \beta = \frac{|\vec{v}|}{c}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Easy way to accomplish task of defining a *Relativistic Mechanics*: write all laws of physics in terms of 4-vectors and 4-tensors, i.e., quantities that transform under Lorentz transformations in the same way as the coordinate differentials.

Four-vectors

Four-vector transformation under z boost Lorentz Transformation

$$v^{0\,\prime} = \gamma\left(v^0 - \beta v^3\right)$$

$$v^{1\,\prime} = v^1$$

$$v^{2\,\prime} = v^2$$

$$v^{3\,\prime} = \gamma\left(v^3 - \beta v^0\right)$$

Important example: Four-velocity. Note that interval

$$d\tau \equiv \sqrt{1-\beta^2}\,dt$$

Lorentz invariant. So the following is a 4-vector

$$cu^\alpha \equiv \left(\frac{dct}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right) = c\gamma\left(1, \beta_x, \beta_y, \beta_z\right)$$

4-Momentum

Single particle mechanics must be defined in terms of Four-momentum

$$p^{\alpha} \equiv mcu^{\alpha} = mc\gamma\left(1, \beta_x, \beta_y, \beta_z\right)$$

Norms, which must be Lorentz invariant, are

$$\sqrt{u_{\alpha}u^{\alpha}} \equiv 1, \sqrt{p_{\alpha}p^{\alpha}} \equiv mc$$

What happens to Newton's Law $\vec{F} = m\vec{a} = d\vec{p}/dt$?

$$\frac{dp^{\alpha}}{d\tau} \equiv F^{\alpha}$$

But need a Four-force on the RHS!!!

Electromagnetic (Lorentz Force)

Non-relativistic

$$\vec{F} = q\left(\vec{E} + \vec{\mathrm{v}} \times \vec{B}\right)$$

Relativistic Generalization (ν summation implied)

$$F^{\alpha} = qF^{\alpha}{}_{\nu}u^{\nu}$$

Electromagnetic Field

$$F^{\alpha}{}_{\nu} \equiv \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{pmatrix}$$

Relativistic Mechanics in E-M Field

Energy Exchange Equation (Note: no magnetic field!)

$$\frac{d\gamma}{dt} = \frac{q\vec{E}\cdot\vec{\mathrm{v}}}{mc^2}$$

Relativistic Lorentz Force Equation (you verify in HW!)

$$\frac{d\left(\gamma m\vec{\mathrm{v}}\right)}{dt} = q\left(\vec{E} + \vec{\mathrm{v}}\times\vec{B}\right)$$

Methods of Acceleration

- Acceleration by Static Electric Fields (DC) Acceleration
  - Cockcroft-Walton
  - van de Graaf Accelerators
  - Limited by voltage breakdowns to potentials of under a million volts in 1930, and presently to potentials of tens of millions of volts (in modern van de Graaf accelerators). Not enough to do nuclear physics at the time.
- Radio Frequency (RF) Acceleration
  - Main means to accelerate in most present day accelerators because one can get to 10-100 MV in a meter these days. Reason: alternating fields don't cause breakdown (if you are careful!) until much higher field levels than DC.
  - Ideas started with Ising and Wideröe

Cockcroft-Walton

Proton Source at Fermilab, Beam Energy 750 keV

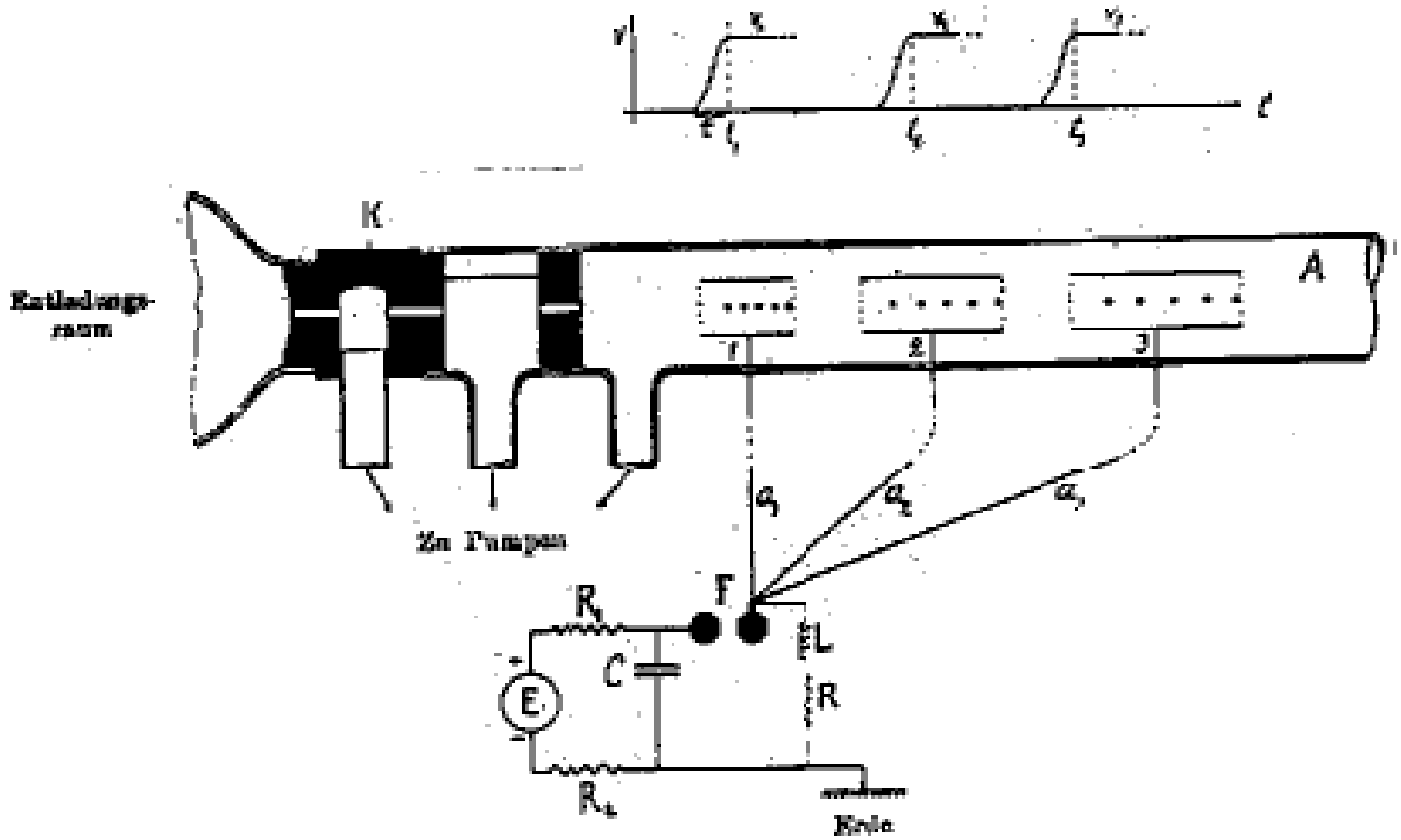
van de Graaf Accelerator

Generator

Brookhaven
Tandem
van de Graaf
~ 15 MV

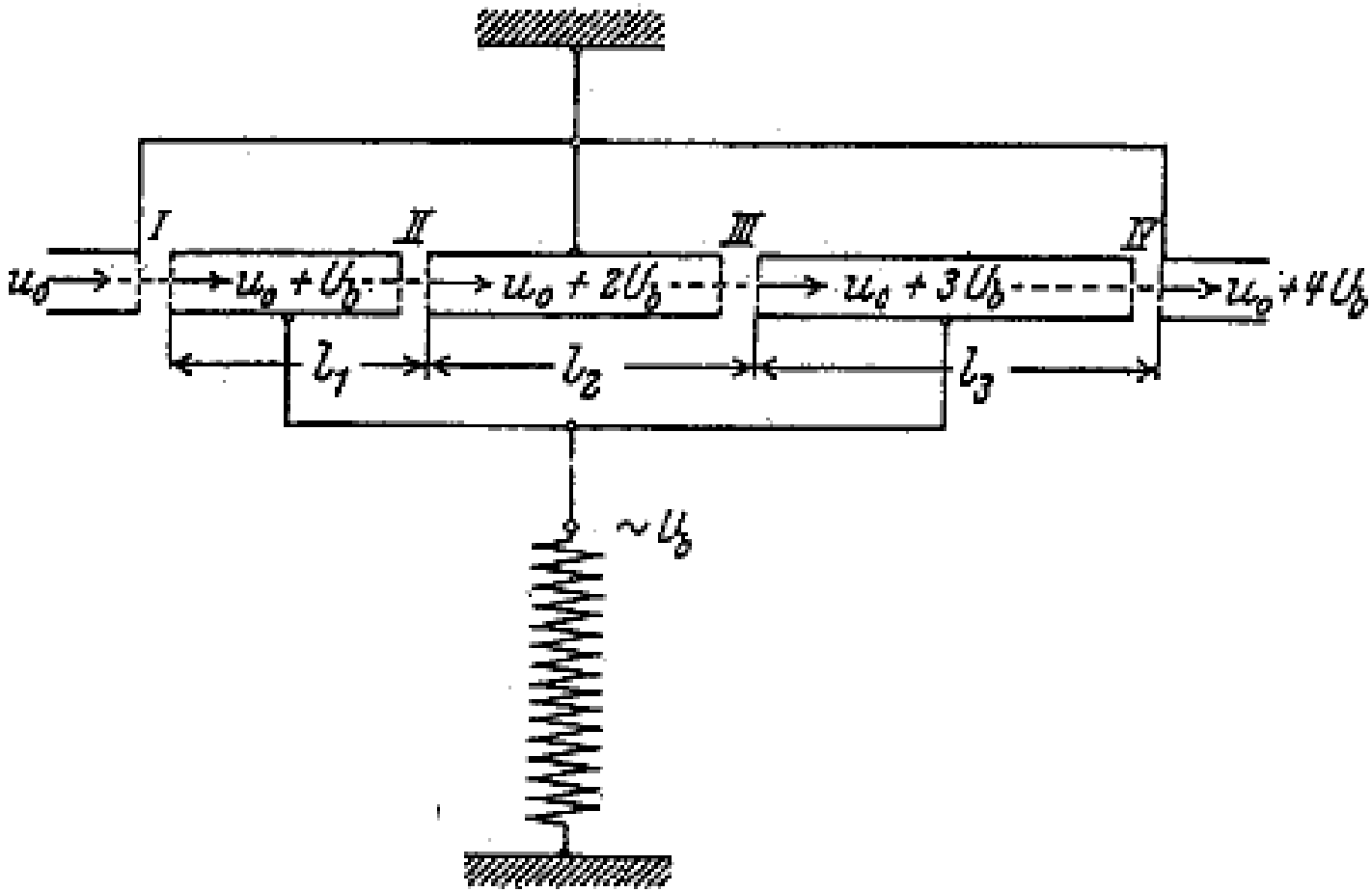
Tandem trick multiplies
the output energy

Ising's Linac Idea



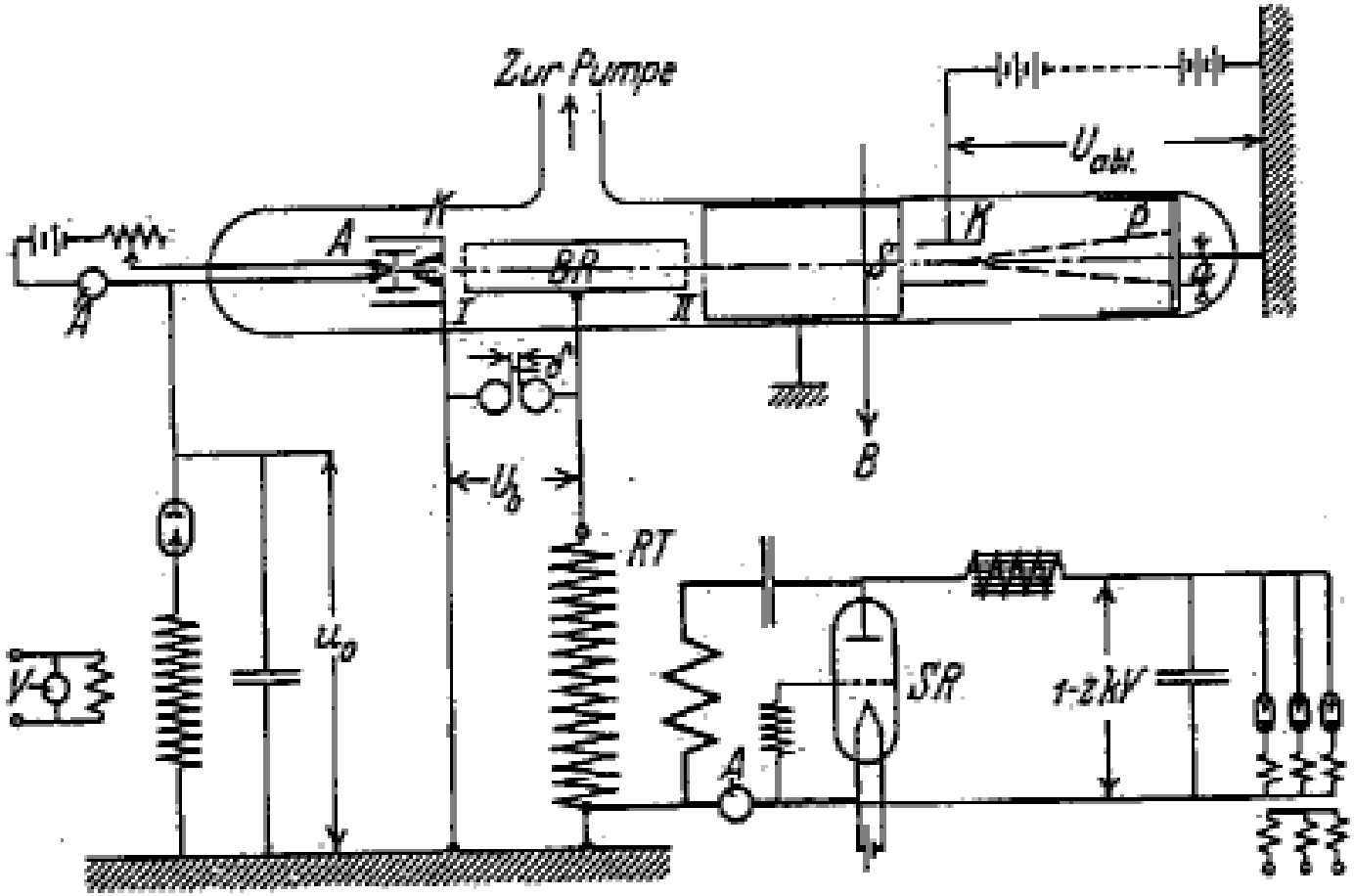
Prinzip einer Methode zur Herstellung von Kanalstrahlen hoher Voltzahl' (in German), Arkiv för matematik o. fysik, 18, Nr. 30, 1-4 (1924).

Drift Tube Linac Proposal



Idea Shown in Wideröe Thesis

Wideröe Thesis Experiment



Über ein neues Prinzip zur Herstellung hoher Spannungen, *Archiv für Elektrotechnik* **21**, 387 (1928)

(On a new principle for the production of higher voltages)

Sloan-Lawrence Heavy Ion Linac

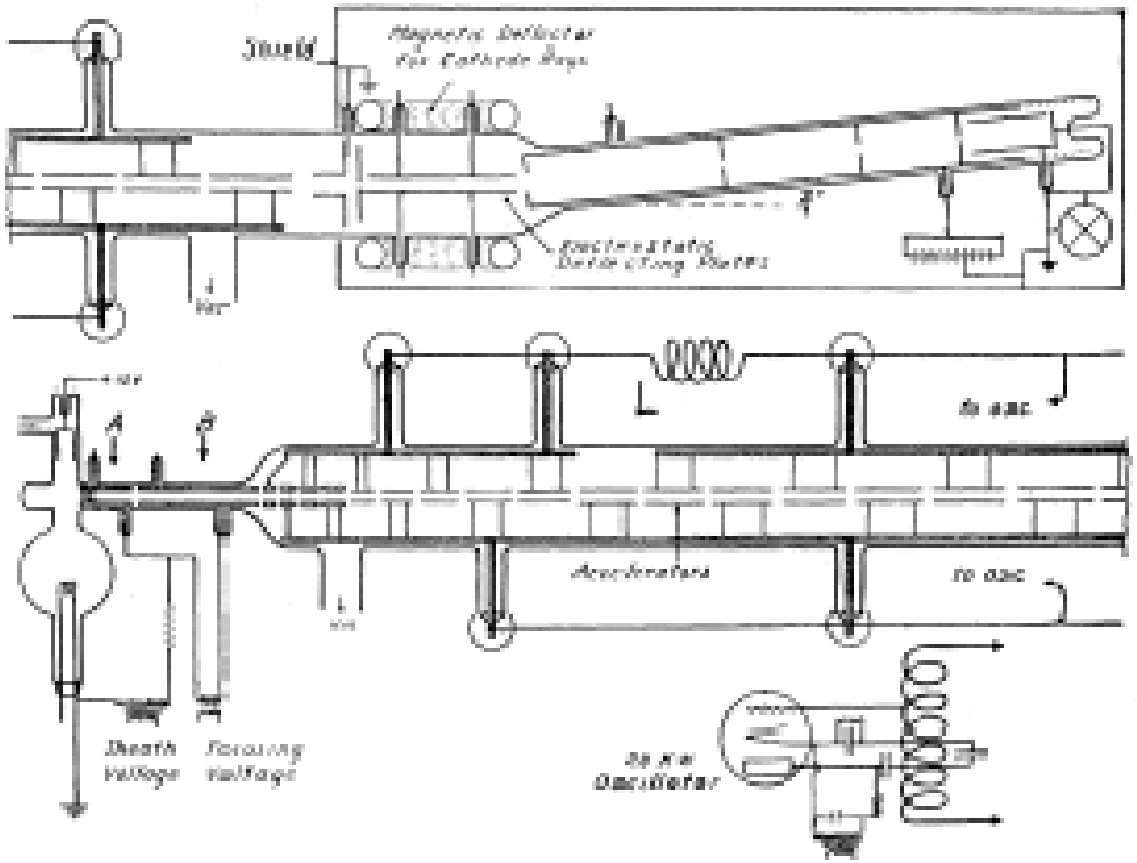


Fig. 1. Diagram of apparatus.

The Production of Heavy High Speed Ions without the Use of High Voltages
David H. Sloan and Ernest O. Lawrence Phys. Rev. **38**, 2021 (1931)

Alvarez Drift Tube Linac

- The first large proton drift tube linac built by Luis Alvarez and Panofsky after WW II

. (1945-1955) Alvarez Proton Linac

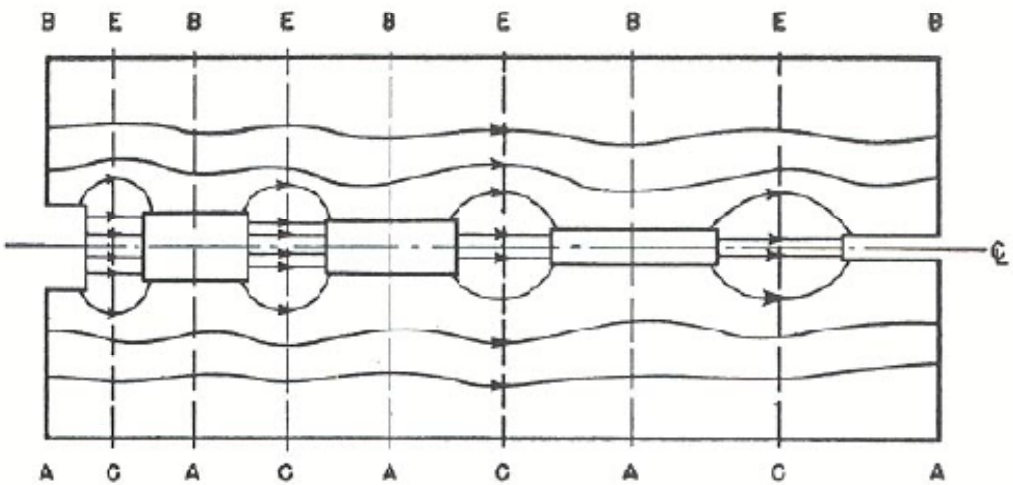
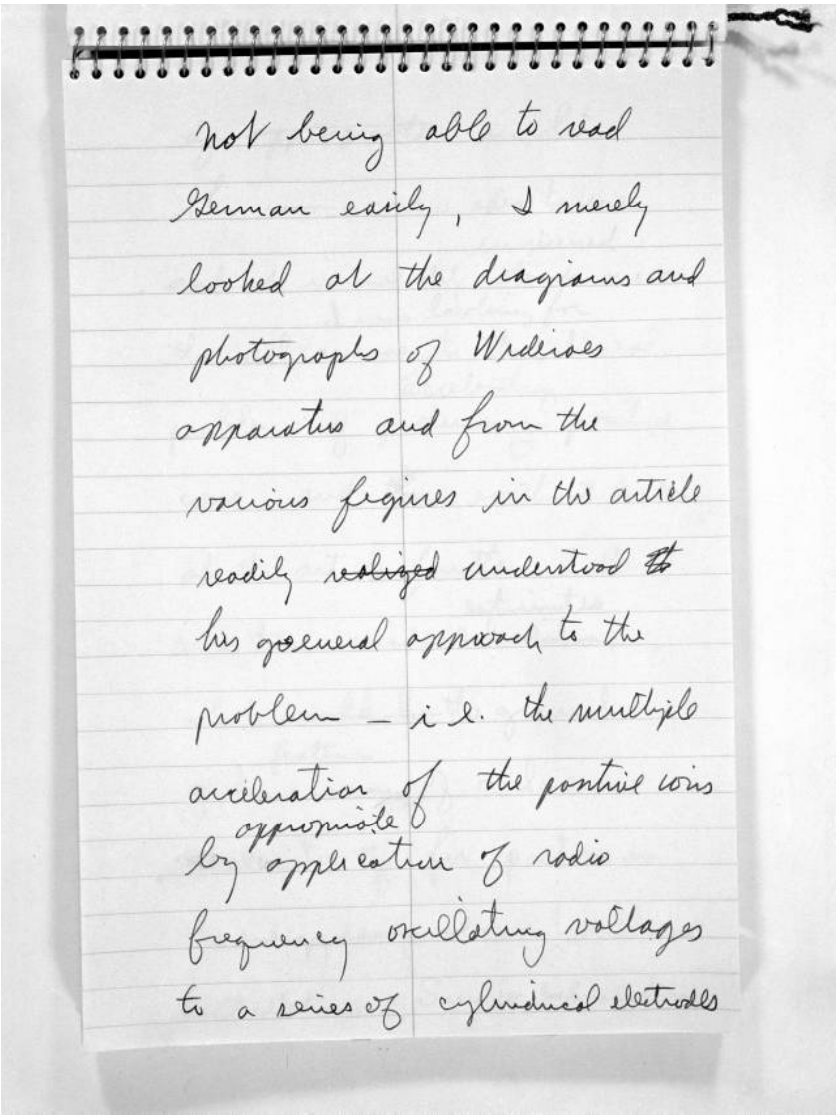


FIG. 2. Linear accelerator produced by introducing drift tubes into cavity excited as in Fig. 1. Division into unit cells.

Alvarez, Bradner, Frank, Gordon, Gow, Marshal, F. Oppenheimer, Panofsky, Richman, and Woodyard, Rev. Sci. Instrum., **26**, 111-133, (1955)

Earnest Orlando Lawrence

Germ of Idea*



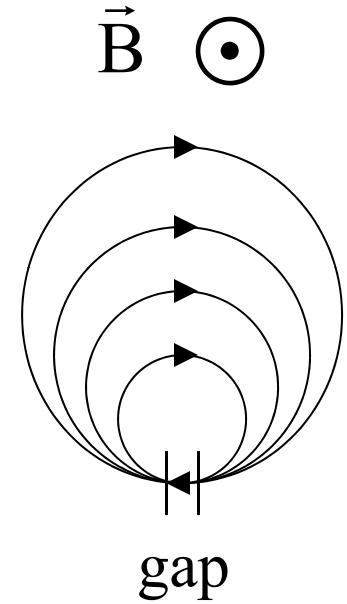
not being able to read
German easily, I merely
looked at the diagrams and
photographs of Wideroes
apparatus and from the
various figures in the article
readily ~~realized~~ understood ~~the~~
his general approach to the
problem – i.e. the multiple
acceleration of the positive ions
by application of appropriate radio
frequency oscillating voltages
to a series of cylindrical electrodes

*Stated in
E. O. Lawrence
Nobel Lecture

Lawrence's Question

- Can you re-use "the same" accelerating gap many times?

$$\vec{F} = m\vec{a} = q\vec{\mathrm{v}} \times \vec{B}$$



$$\frac{d^2x}{dt^2} = \frac{qB}{m}\mathrm{v}_y \quad \rightarrow \frac{d^2\mathrm{v}_x}{dt^2} + \Omega_c^2 \mathrm{v}_x = 0$$

$$\frac{d^2y}{dt^2} = -\frac{qB}{m}\mathrm{v}_x \rightarrow \frac{d^2\mathrm{v}_y}{dt^2} + \Omega_c^2 \mathrm{v}_y = 0$$

$$\frac{d}{dt}\left(\mathrm{v}_x^2 + \mathrm{v}_y^2\right) = \frac{qB}{m}\left(\mathrm{v}_x\mathrm{v}_y - \mathrm{v}_y\mathrm{v}_x\right) = 0$$

$\mathrm{v}_0 = \sqrt{\mathrm{v}_x^2(t) + \mathrm{v}_y^2(t)}$ is a constant of the motion

Cyclotron Frequency

$$\mathrm{v}_x(t) = \mathrm{v}_0 \cos(\Omega_c t + \delta); \mathrm{v}_y(t) = -\mathrm{v}_0 \sin(\Omega_c t + \delta)$$

$$x(t) = x_0 + \frac{\mathrm{v}_0}{\Omega_c} \sin(\Omega_c t + \delta); y(t) = y_0 + \frac{\mathrm{v}_0}{\Omega_c} \cos(\Omega_c t + \delta)$$

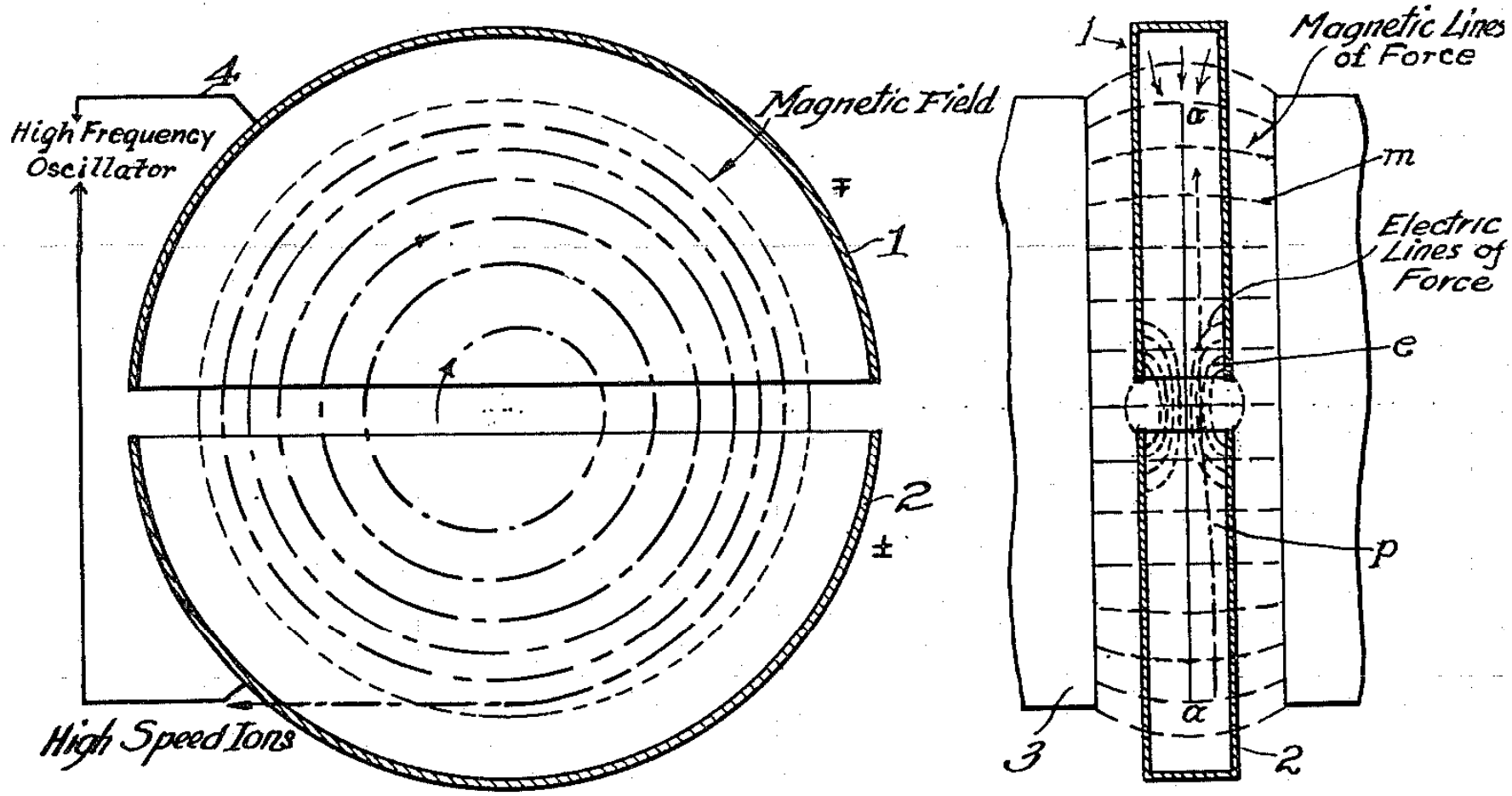
The radius of the oscillation $r = \mathrm{v}_0/\Omega_c$ is proportional to the velocity after the gap. Therefore, the particle takes the same amount of time to come around to the gap, independent of the actual particle energy!!!! (only in the non-relativistic approximation). Establish a resonance (equality!) between RF frequency and particle transverse oscillation frequency, also known as the Cyclotron Frequency

$$f_{rf} = f_c = \Omega_c / 2\pi = \frac{qB}{2\pi m}$$

What Correspond to Drift Tubes?

- Dee's!

U. S. Patent Diagram

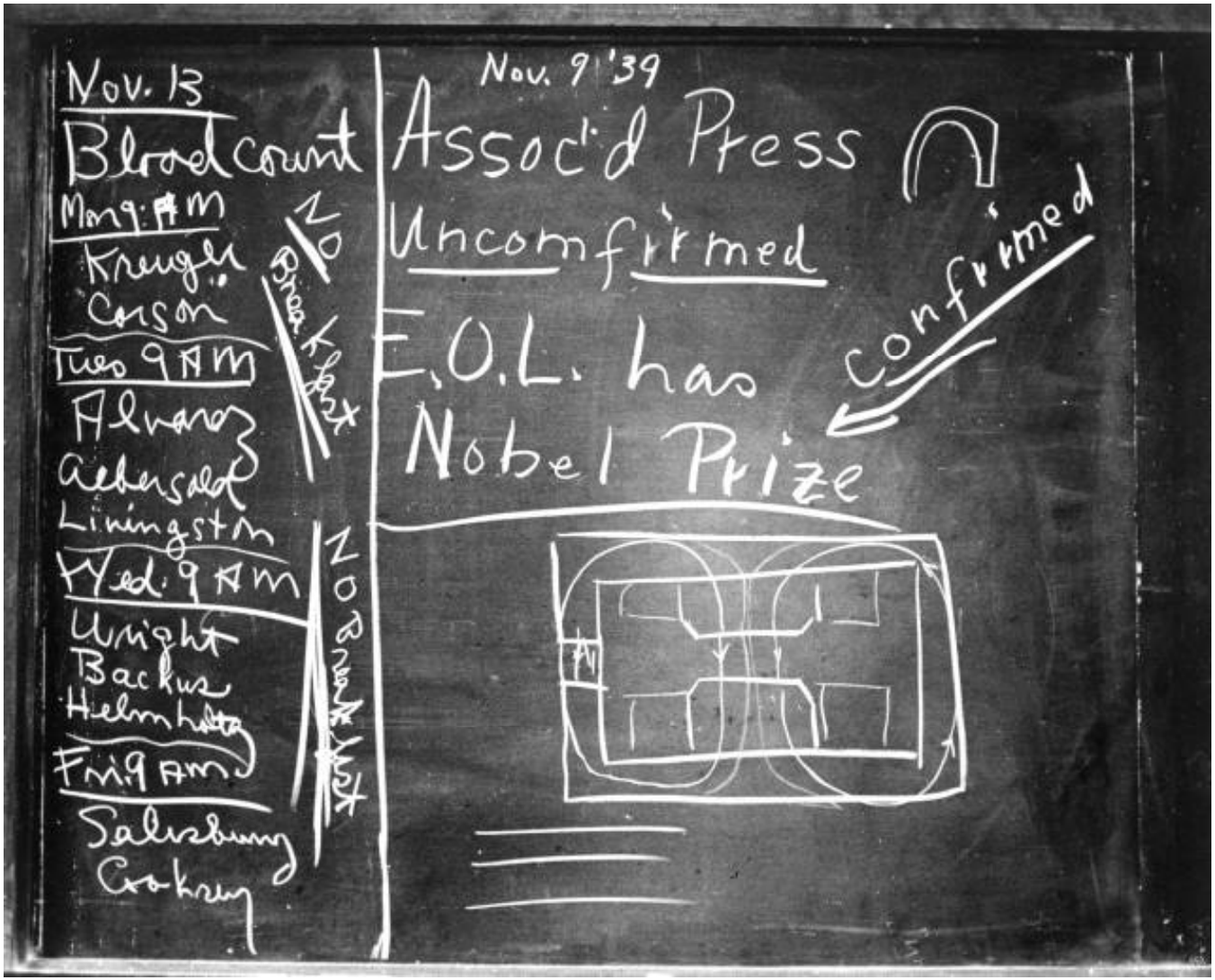


Magnet for 27 Inch Cyclotron (LHS)

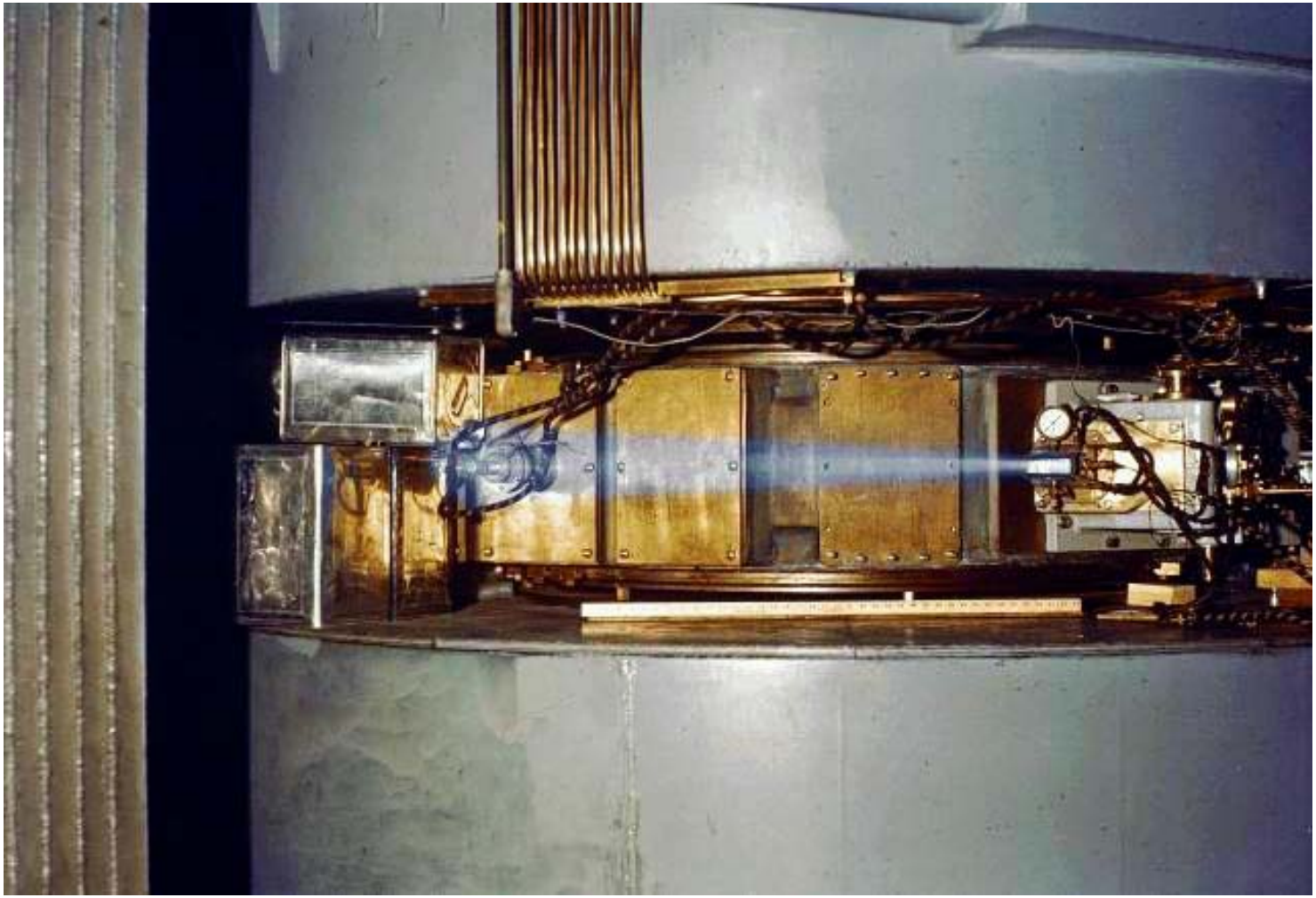
Lawrence and "His Boys"

1-4: Jack Livingood, Frank Exner, M.S.Livingston, David Sloan, E.O.Lawrence, Milton White, Wesley Coates, L.Jackson Laslett and Commander T. Lucci - 1933

And Then!

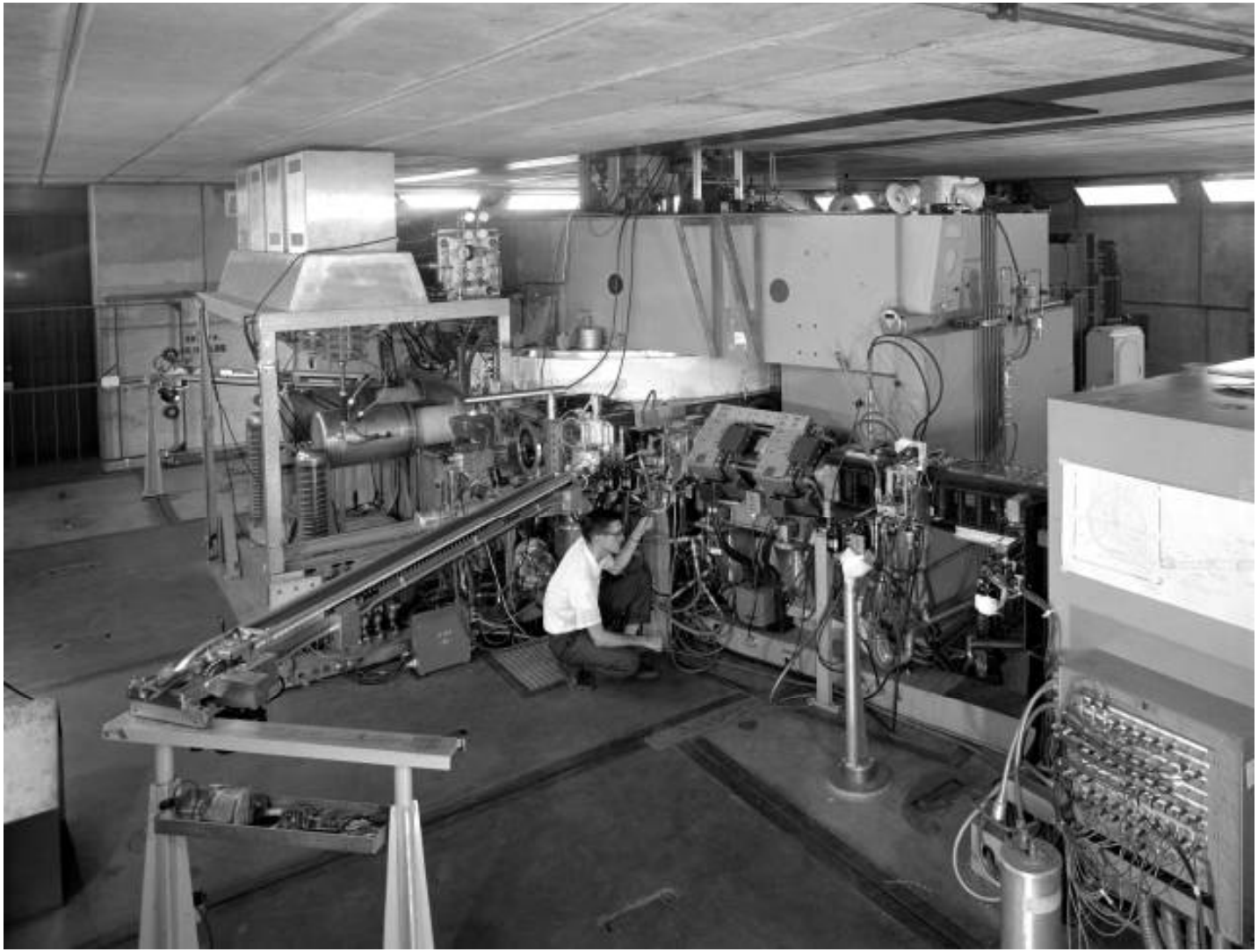

Nov. 13
Blood count
Mon 9 AM
Kruger
Carson
Tues 9 AM
Alvarez
Oebersold
Livingston
Wed. 9 AM
Wright
Backus
Helmholtz
Fri 9 AM
Salisbury
Cockroft
No Break fast
No Breakfast
Nov. 9 '39
Associ'd Press
Uncomfirmed
E.O.L. has
Nobel Prize
confirmed

Beam Extracted from a Cyclotron



Radiation Laboratory 60 Inch Cyclotron, circa 1939

88 Inch Cyclotron at Berkeley Lab



Relativistic Corrections

When include relativistic effects (you'll see in the HW!) the "effective" mass to compute the oscillation frequency is the relativistic mass γm

$$f_c = \Omega_c / 2\pi = \frac{qB}{2\pi\gamma m}$$

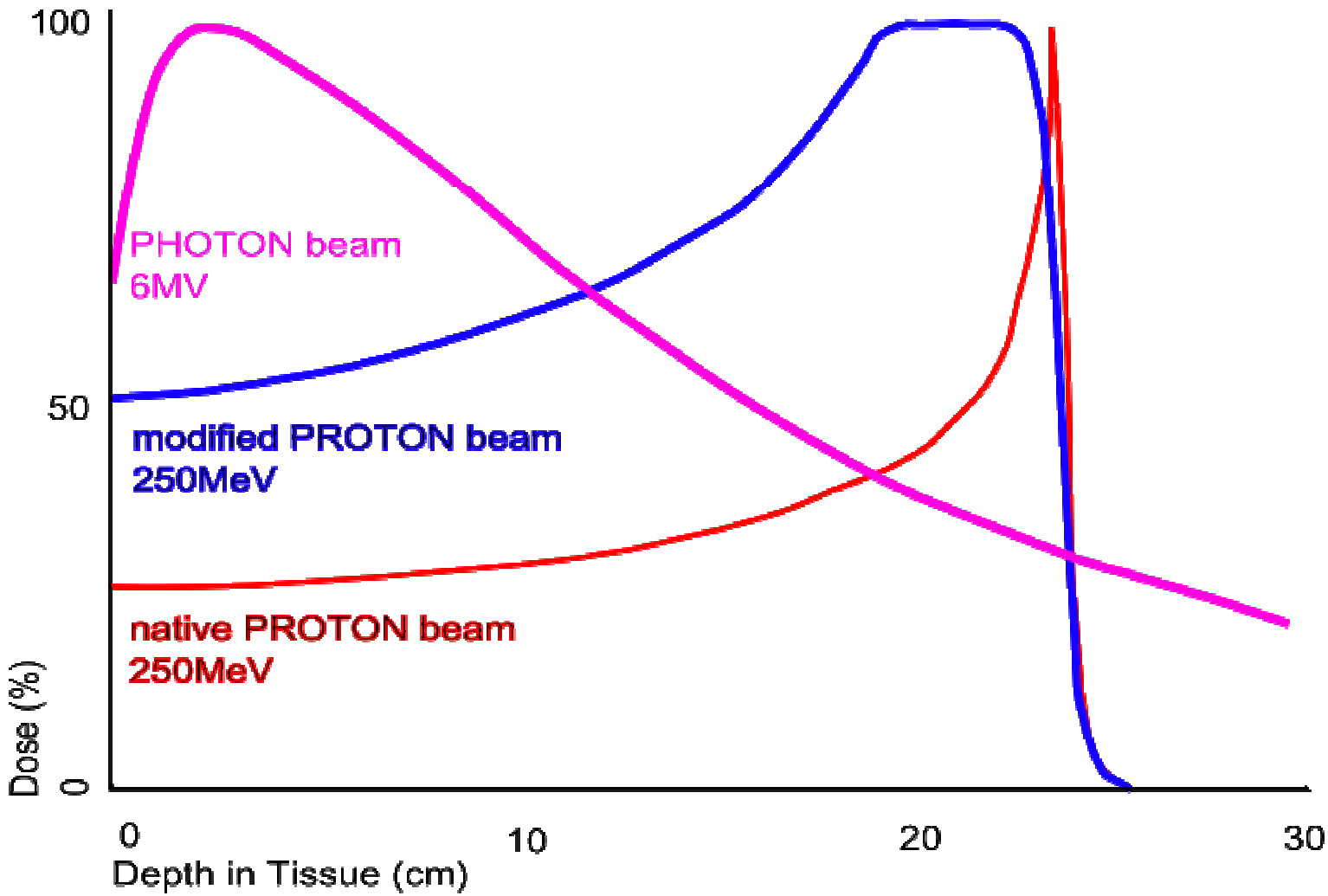
where γ is Einstein's relativistic γ, most usefully expressed as

$$\gamma = \frac{E_{tot}}{E_0} = \frac{E_0 + E_{kin}}{E_0} = \frac{mc^2 + E_{kin}}{mc^2}$$

m particle rest mass, E_{kin} particle kinetic energy

Cyclotrons for Radiation Therapy

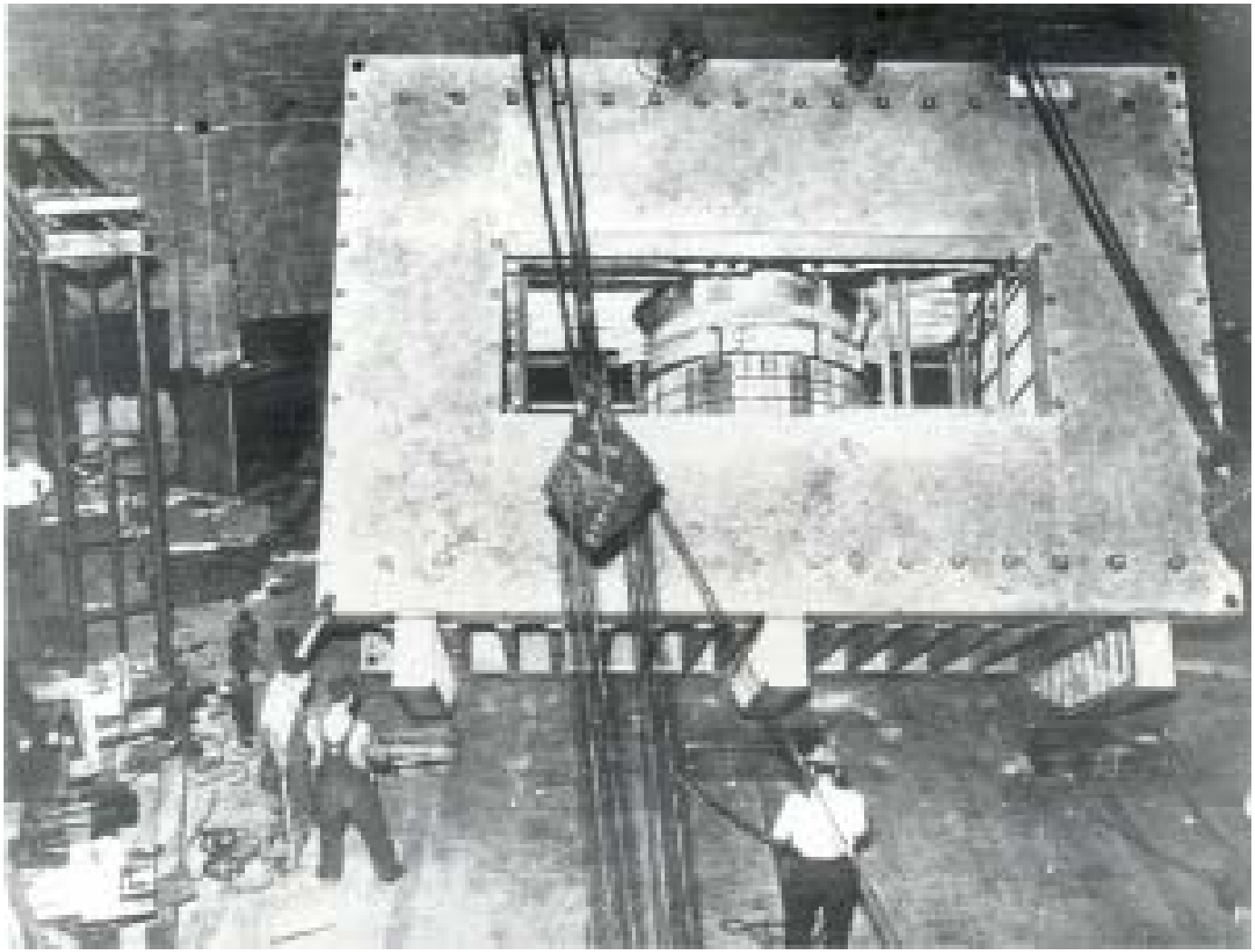
Bragg Peak


100
50
0
Dose (%)
PHOTON beam
6MV
modified PROTON beam
250MeV
native PROTON beam
250MeV
0
10
20
30
Depth in Tissue (cm)

Betatrons

25 MeV electron accelerator with its inventor: Don Kerst. The earliest electron accelerators for medical uses were betatrons.

300 MeV ~ 1949



Electromagnetic Induction

Faraday's Law: Differential Form of Maxwell Equation

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

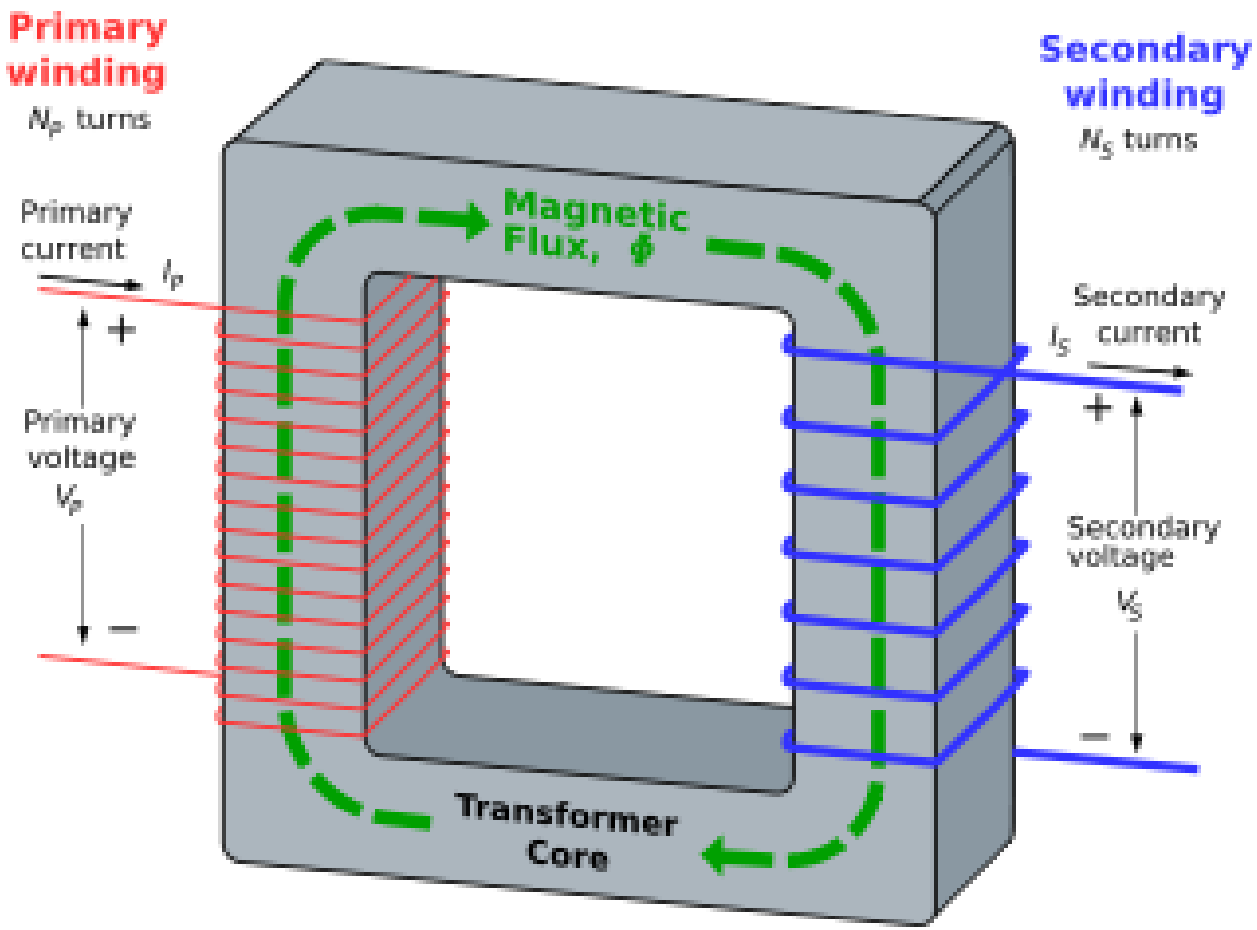
Faraday's Law: Integral Form

$$\oint_S \vec{\nabla} \times \vec{E} \cdot d\vec{S} = -\oint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

Faraday's Law of Induction

$$\oint_{\partial S} \vec{E} \cdot d\vec{l} = 2\pi R E_\theta = -\frac{d}{dt}\Phi_B$$

Transformer



Betatron as a Transformer

- In the betatron the electron beam itself is the secondary winding of the transformer. Energy transferred directly to the electrons

$$2\pi R E_\theta = -\frac{d}{dt}\Phi_B$$

- Radial Equilibrium

$$R = \frac{\beta c}{eB/\gamma m}$$

- Energy Gain Equation

$$\frac{d\gamma}{dt} = \frac{eE_\theta \beta c}{mc^2}$$

Betatron condition

To get radial stability in the electron beam orbit (i.e., the orbit radius does not change during acceleration), need

$$R = const \Rightarrow \frac{dB}{dt} = \frac{B}{\gamma}\frac{d\gamma}{dt} \text{ and } \frac{B}{\gamma} \approx \frac{cm}{eR}$$

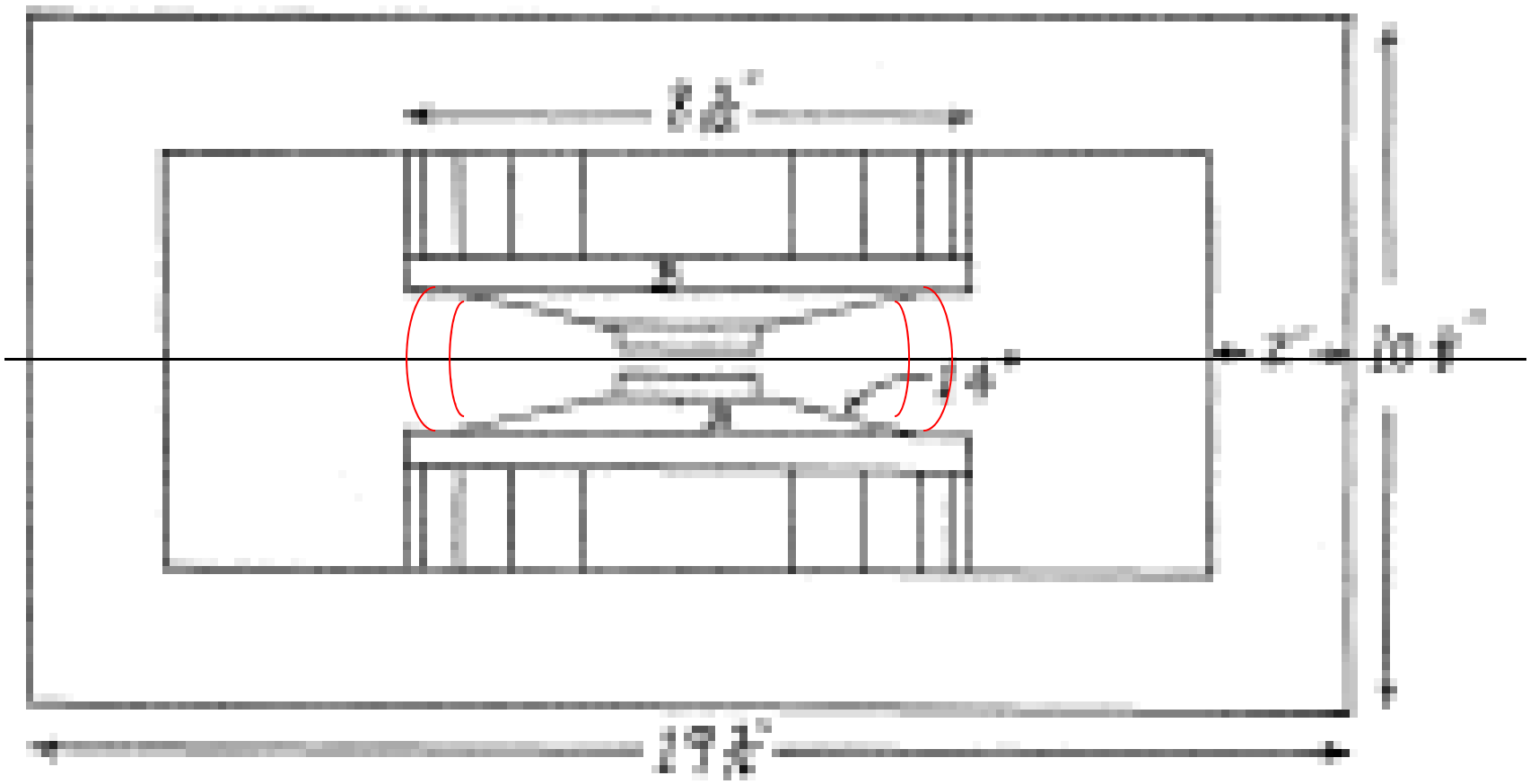
$$\Phi_B = \alpha \pi R^2 B \text{ for some } \alpha \text{ and } \frac{d\gamma}{dt} \approx \frac{ec}{mc^2}\frac{1}{2\pi R}\frac{d\Phi_B}{dt} \Rightarrow \alpha = 2$$

$$\therefore \Phi_B = 2\pi R^2 B\left(r = R\right)$$

This last expression is sometimes called the "betatron two for one" condition. The energy increase from the flux change is

$$\gamma - \gamma_0 \approx \frac{q\beta c}{2\pi R mc^2}\Delta\Phi_B$$

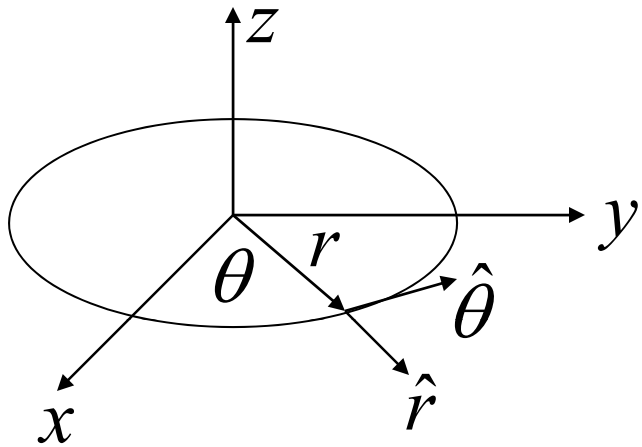
Transverse Beam Stability



Ensured by proper shaping of the magnetic field in the betatron

Relativistic Equations of Motion

Standard Cylindrical Coordinates



$$\frac{d\vec{\mathrm{v}}}{dt} = \frac{q}{\gamma m}\left(\vec{\mathrm{v}} \times \vec{B}\right) \qquad \frac{d\gamma}{dt} = 0!!$$

$$r^2 = x^2 + y^2$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y} \qquad \hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

$$\mathrm{v}_r = \vec{\mathrm{v}} \cdot \hat{r} = \dot{r} \qquad \mathrm{v}_\theta = \vec{\mathrm{v}} \cdot \hat{\theta} = r\dot{\theta}$$

$$\frac{d\vec{\mathrm{v}}}{dt} = \frac{d}{dt}\left(\mathrm{v}_r\hat{r} + \mathrm{v}_\theta\hat{\theta}\right)$$

$$d\hat{r}/dt = \dot{\theta}\hat{\theta}$$

$$d\hat{\theta}/dt = -\dot{\theta}\hat{r}$$

Cylindrical Equations of Motion

In components

$$\ddot{r} - r\dot{\theta}^2 = \frac{q}{\gamma m}\left(\vec{\mathrm{v}} \times \vec{B}\right)_r = \frac{q}{\gamma m} r\dot{\theta} B_z$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{q}{\gamma m}\left(\vec{\mathrm{v}} \times \vec{B}\right)_\theta = \frac{q}{\gamma m}\left(\dot{z}B_r - \dot{r}B_z\right)$$

$$\ddot{z} = \frac{q}{\gamma m}\left(\vec{\mathrm{v}} \times \vec{B}\right)_z = -\frac{q}{\gamma m} r\dot{\theta} B_r$$

Zero'th order solution

$$r(t) = cons = R$$

$$\theta(t) = \theta_0 + \dot{\theta}_0 t \qquad z(t) = 0$$

Magnetic Field Near Orbit

Get cyclotron frequency again, as should

$$\dot{\theta}_0 = -\frac{qB_z\left(r = R, z = 0\right)}{\gamma m} = \Omega_c$$

Magnetic field near equilibrium orbit

$$\vec{B}\left(r,z\right) \square\ B_0\hat{z} + \frac{\partial B_r}{\partial r}\left(r - R\right)\hat{r} + \frac{\partial B_z}{\partial r}\left(r - R\right)\hat{z} +$$

$$\frac{\partial B_r}{\partial z} z\hat{r} + \frac{\partial B_z}{\partial z} z\hat{z}$$

$$\vec{\nabla} \times \vec{B} = 0 \rightarrow \frac{\partial B_z}{\partial r} = \frac{\partial B_r}{\partial z} \qquad \vec{\nabla} \cdot \vec{B} = 0, B_r = 0 \rightarrow \frac{\partial B_z}{\partial z} = 0$$

Field Index

Magnetic Field completely specified by its z-component on the mid-plane

$$\vec{B}(r,z) \approx B_0\hat{z} + \frac{\partial B_z}{\partial r}\left[(r-R)\hat{z} + z\hat{r}\right]$$

Power Law model for fall-off

$$B_z(r, z=0) \approx B_0 (R/r)^n$$

The constant n describing the falloff is called the field index

$$\vec{B}(r,z) \approx B_0\hat{z} - \frac{nB_0}{R}\left[(r-R)\hat{z} + z\hat{r}\right]$$

Linearized Equations of Motion

Assume particle orbit "close to" or "nearby" the unperturbed orbit

$$\delta r(t) = r(t) - R \qquad \delta\theta(t) = \theta(t) - \Omega_c t \qquad \delta z(t) = z(t)$$

$$B_z \approx B_0 - \frac{nB_0}{R}\delta r \qquad B_r \approx -\frac{nB_0}{R}\delta z$$

$$\delta\ddot{r} - \delta r\Omega_c^2 - 2R\Omega_c\delta\dot{\theta} = \frac{q}{\gamma m}\left[\delta r\Omega_c B_0 + R\delta\dot{\theta}B_0 - R\Omega_c\frac{nB_0}{R}\delta r\right]$$

$$R\delta\ddot{\theta} + 2\delta\dot{r}\Omega_c = \delta\dot{r}\Omega_c \rightarrow R\delta\dot{\theta} + \delta r\Omega_c = const$$

$$\delta\ddot{z} = \frac{q}{\gamma m}R\Omega_c\frac{nB_0}{R}\delta z = -n\Omega_c^2\delta z$$

"Weak" Focusing

For small deviations from the unperturbed circular orbit the transverse deviations solve the (driven!) harmonic oscillator equations

$$\delta\ddot{r} + \left(1-n\right)\Omega_c^2\delta r = \Omega_c const$$

$$\delta\ddot{z} + n\Omega_c^2\delta z = 0$$

The small deviations oscillate with a frequency $n^{1/2}\Omega_c$ in the vertical direction and $(1-n)^{1/2}\,\Omega_c$ in the radial direction. Focusing by magnetic field shaping of this sort is called Weak Focusing. This method was the primary method of focusing in accelerators up until the mid 1950s, and is still occasionally used today.

Stability of Transverse Oscillations

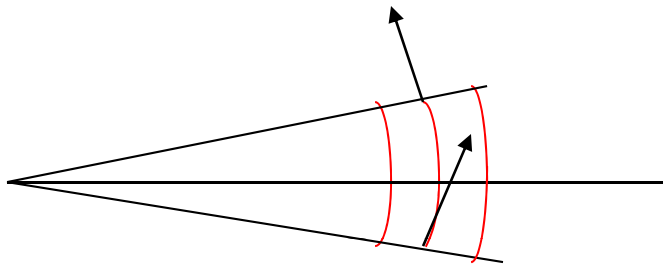
- For long term stability, the field index must satisfy

$$0 < n < 1$$

because only then do the transverse oscillations remain bounded for all time. Because transverse oscillations in accelerators were theoretically studied by Kerst and Serber (*Physical Review,* **60**, 53 (1941)) for the first time in betatrons, transverse oscillations in accelerators are known generically as betatron oscillations. Typically n was about 0.6 in betatrons.

Physical Source of Focusing

$0 < n$



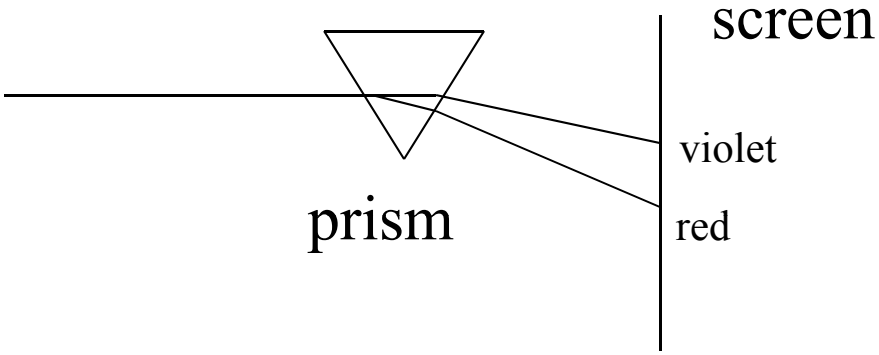
B_r changes sign as go through mid-plane. B_z weaker as r increases

$n < 1$

Bending on a circular orbit is naturally focusing in the bend direction (why?!), and accounts for the 1 in $1 - n$. Magnetic field gradient that causes focusing in z causes defocusing in r, essentially because $\partial B_z / \partial r = \partial B_r / \partial z$. For $n > 1$, the defocusing wins out.

First Look at Dispersion

Newton's Prism Experiment

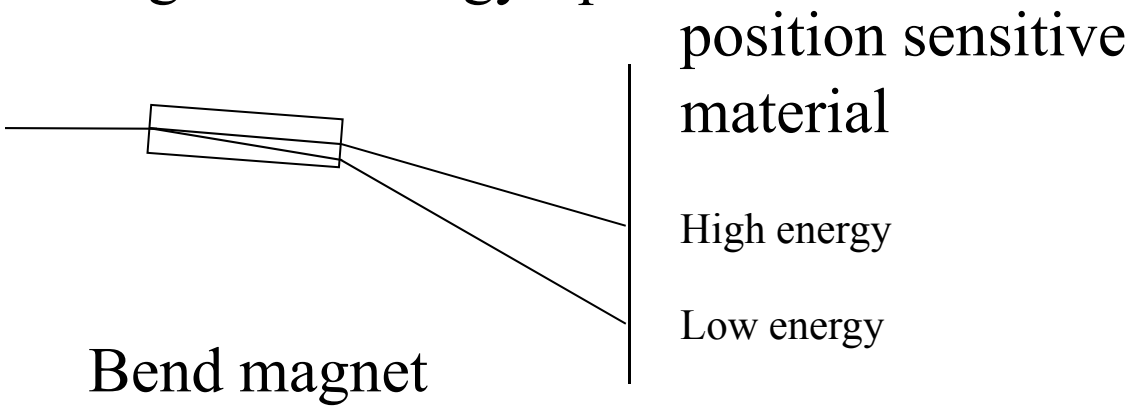


$$\Delta x = D\left(\frac{\Delta p}{p}\right)$$

$$\Delta x = \eta\left(\frac{\Delta p}{p}\right)$$

Dispersion units: m

Bend Magnet as Energy Spectrometer



Dispersion for Betatron

Radial Equilibrium

$$R = \frac{\beta c}{eB/\gamma m} = \frac{p}{eB}$$

Linearized

$$\left(R + \Delta R\right)\left(B_0 + \Delta B\right) = \frac{p + \Delta p}{e} \approx RB_0 + R\Delta B + \Delta RB_0$$

$$\frac{\Delta p}{e} \approx -n\Delta RB_0 + \Delta RB_0 = \left(1 - n\right)\Delta RB_0$$

$$\frac{\Delta p}{p} \approx \left(1 - n\right)\frac{\Delta R}{R} \rightarrow D_{radial} = \frac{R}{\left(1 - n\right)}$$

Evaluate the constant

$$\delta\ddot{r} + \left(1-n\right)\Omega_c^2\delta r = \Omega_c const$$

For a time independent solution $\delta r = \Delta R$ (orbit at larger radius)

$$\left(1-n\right)\Omega_c^2\Delta R = \Omega_c const$$

$$const = \left(1-n\right)\Omega_c D_{radial}\frac{\Delta p}{p} = \Omega_c R\frac{\Delta p}{p}$$

General Betatron Oscillation equations

$$\delta\ddot{r} + \left(1-n\right)\Omega_c^2\delta r = \Omega_c^2 R\frac{\Delta p}{p}$$

$$\delta\ddot{z} + n\Omega_c^2\delta z = 0$$

No Longitudinal Focusing

$$R\delta\dot{\theta} + \Omega_c \delta r = \Omega_c R \frac{\Delta p}{p}$$

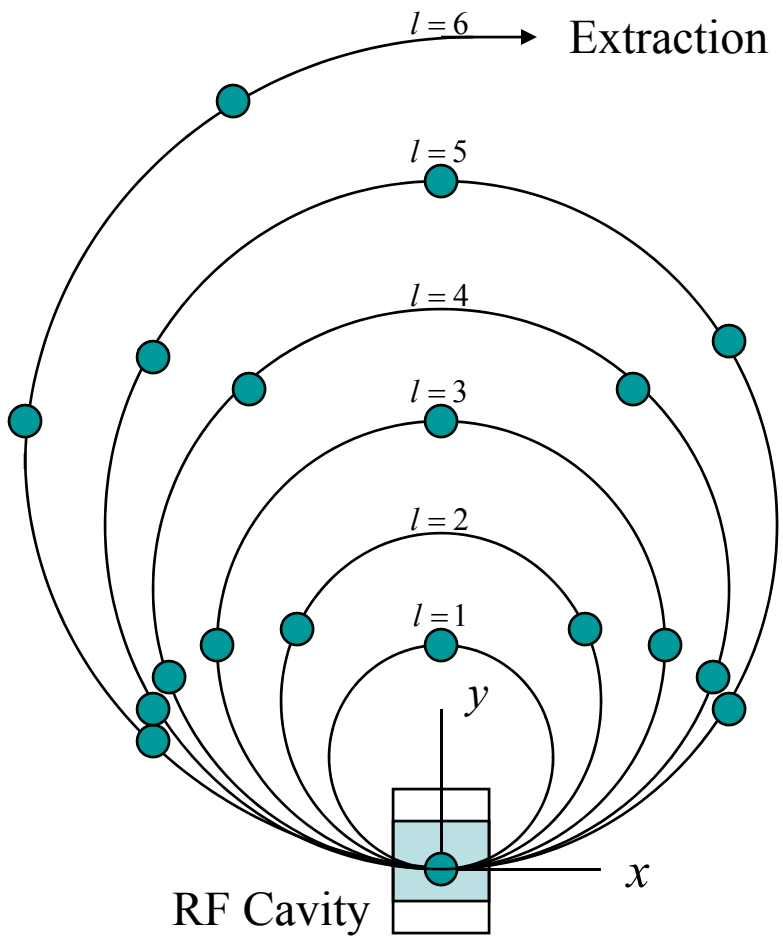
$$\theta = \theta_0 + \Omega_c t + \int \left[\Omega_c \frac{\Delta p}{p} - \Omega_c \frac{\Delta R}{R} \right] dt$$

$$= \theta_0 + \Omega_c t + \int \Omega_c \frac{\Delta p}{p} \left[1 - \frac{1}{1-n} \right] dt$$

Greater Speed

Weaker Field

Classical Microtron: Veksler (1945)



$\otimes$ Magnetic Field

$\mu = 2$

$\nu = 1$

Basic Principles

For the geometry given

$$\frac{d(\gamma m \vec{\mathrm{v}})}{dt} = -e\left[\vec{E} + \vec{\mathrm{v}} \times \vec{B}\right]$$

$$\frac{d(\gamma m \mathrm{v}_x)}{dt} = e\mathrm{v}_y B_z$$

$$\frac{d(\gamma m \mathrm{v}_y)}{dt} = -e\mathrm{v}_x B_z$$

$$\frac{d^2\mathrm{v}_x}{dt^2} + \frac{\Omega_c^2}{\gamma^2}\mathrm{v}_x = 0 \qquad \frac{d^2\mathrm{v}_y}{dt^2} + \frac{\Omega_c^2}{\gamma^2}\mathrm{v}_y = 0$$

For each orbit, separately, and exactly

$$\mathrm{v}_x(t) = -v_{x0}\cos(\Omega_c t/\gamma) \qquad \mathrm{v}_y(t) = v_{x0}\sin(\Omega_c t/\gamma)$$

$$x(t) = -\frac{\gamma v_{x0}}{\Omega_c}\sin\left(\Omega_c t/\gamma\right) \qquad y(t) = \frac{\gamma v_{x0}}{\Omega_c} - \frac{\gamma v_{x0}}{\Omega_c}\cos\left(\Omega_c t/\gamma\right)$$

Non-relativistic cyclotron frequency: $\Omega_c = 2\pi f_c = eB_z / m$

Relativistic cyclotron frequency: Ω_c / γ

Bend radius of each orbit is: $\rho_l = \gamma_l \mathrm{v}_{x0,l} / \Omega_c \rightarrow \gamma_l c / \Omega_c$

In a conventional cyclotron, the particles move in a circular orbit that grows in size with energy, but where the relatively heavy particles stay in resonance with the RF, which drives the accelerating DEEs at the non-relativistic cyclotron frequency. By contrast, a microtron uses the "other side" of the cyclotron frequency formula. The cyclotron frequency decreases, proportional to energy, and the beam orbit radius increases in each orbit by precisely the amount which leads to arrival of the particles in the succeeding orbits precisely in phase.

Microtron Resonance Condition

Must have that the bunch pattern repeat in time. This condition is only possible if the time it takes to go around each orbit is precisely an integral number of RF periods

$$\gamma_1 = \mu \frac{f_c}{f_{RF}}$$

First Orbit

$$\Delta\gamma = \nu \frac{f_c}{f_{RF}}$$

Each Subsequent Orbit

For classical microtron assume can inject so that

$$\gamma_1 \approx 1 + \nu \frac{f_c}{f_{RF}}$$

$$\frac{f_c}{f_{RF}} \approx \frac{1}{\mu - \nu}$$

Parameter Choices

The energy gain in each pass must be identical for this resonance to be achieved, because once f_c/f_{RF} is chosen, $\Delta\gamma$ is fixed. Because the energy gain of non-relativistic ions from an RF cavity IS energy dependent, there is no way (presently!) to make a classical microtron for ions. For the same reason, in electron microtrons one would like the electrons close to relativistic after the first acceleration step. Concern about injection conditions which, as here in the microtron case, will be a recurring theme in examples!

$$f_c / f_{RF} = B_z / B_0 \qquad B_0 = \frac{2\pi mc}{\lambda e}$$

$$B_0 = 0.107\ \mathrm{T} = 1.07\ \mathrm{kG@10cm}$$

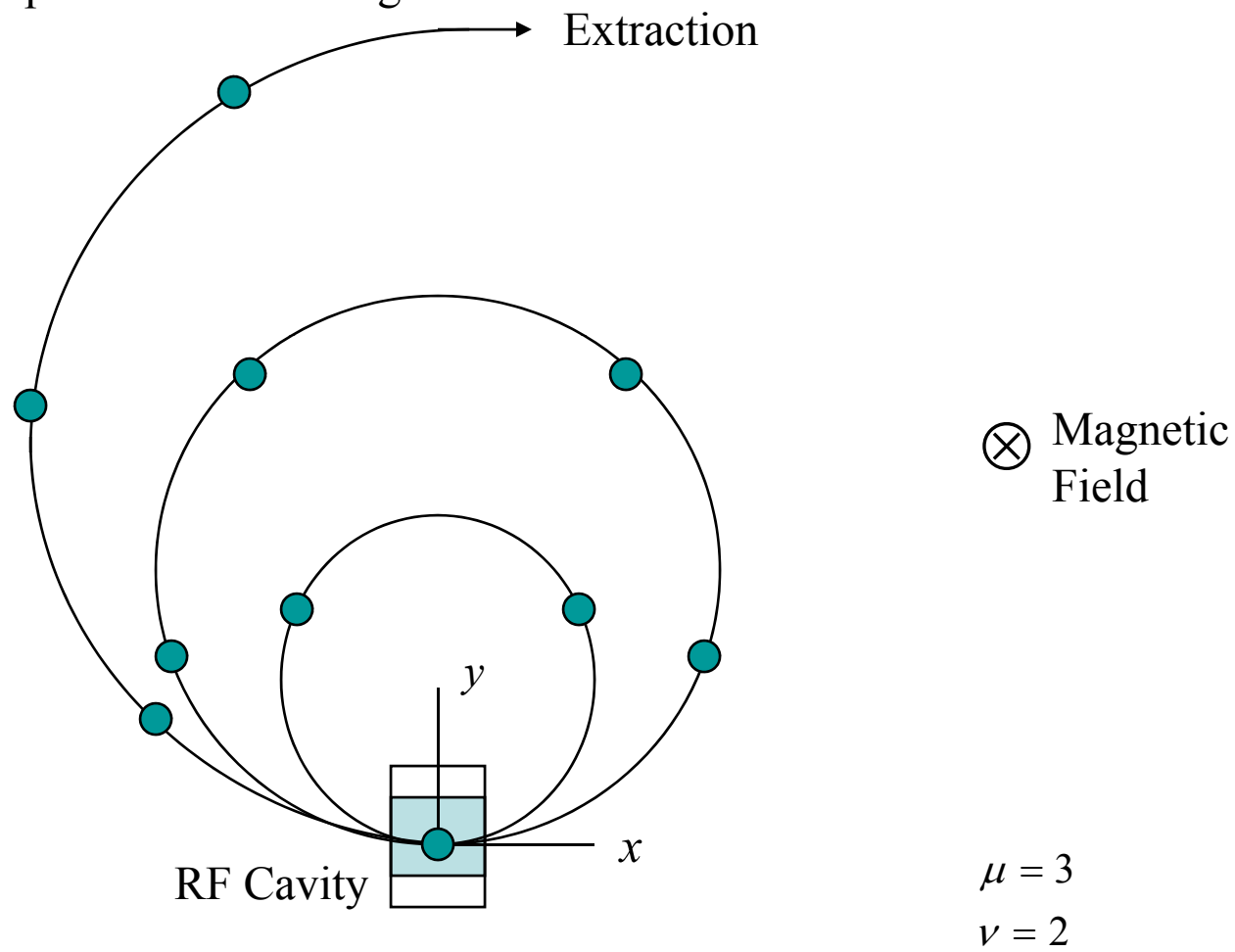
Notice that this field strength is NOT state-of-the-art, and that one normally chooses the magnetic field to be around this value. High frequency RF is expensive too!

Classical Microtron Possibilities

Assumption: Beam injected at low energy and energy gain is the same for each pass

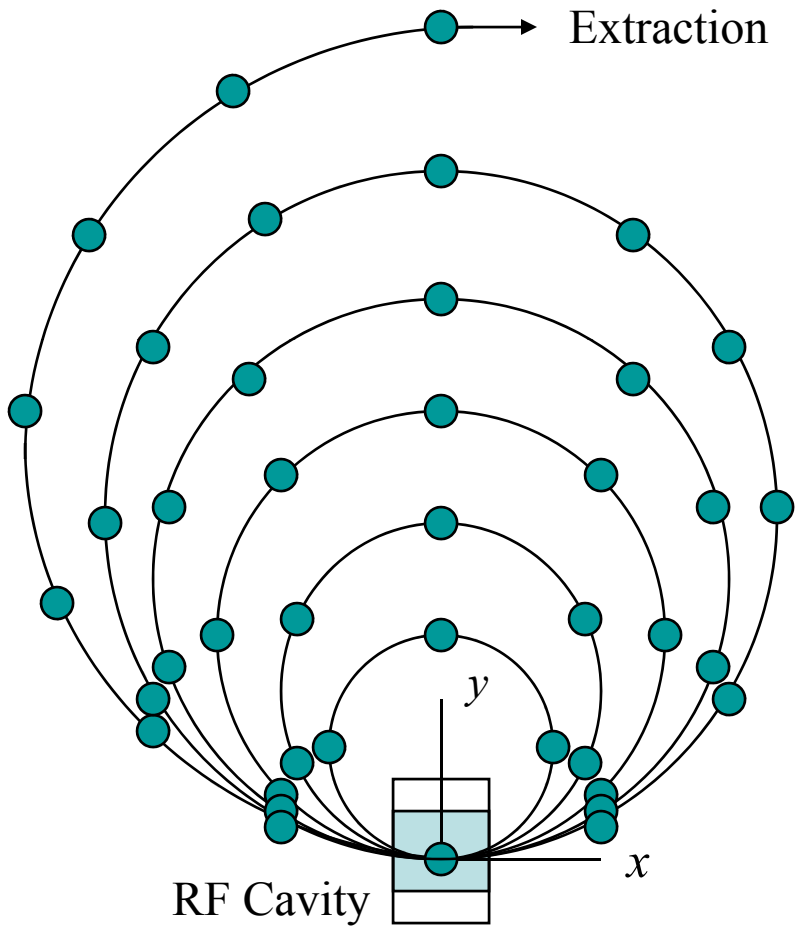
$\frac{f_c}{f_{RF}}$	1	1/2	1/3	1/4	
	$\mu, \nu, \gamma_1, \Delta\gamma$	$\mu, \nu, \gamma_1, \Delta\gamma$	$\mu, \nu, \gamma_1, \Delta\gamma$	$\mu, \nu, \gamma_1, \Delta\gamma$	...
	2, 1, 2, 1	3, 1, 3/2, 1	4, 1, 4/3, 1	5, 1, 5/4, 1	...
	3, 2, 3, 2	4, 2, 2, 2	5, 2, 5/3, 2	6, 2, 3/2, 2	...
	4, 3, 4, 3	5, 3, 5/2, 3	6, 3, 2, 3	7, 3, 7/4, 3	...
	5, 4, 5, 4	6, 4, 3, 4	7, 4, 7/3, 4	8, 4, 2, 4	...
	⋮	⋮	⋮	⋮	⋱

For same microtron magnet, no advantage to higher n; RF is more expensive because energy per pass needs to be higher



Going along diagonal changes frequency

To deal with lower frequencies, go up the diagonal



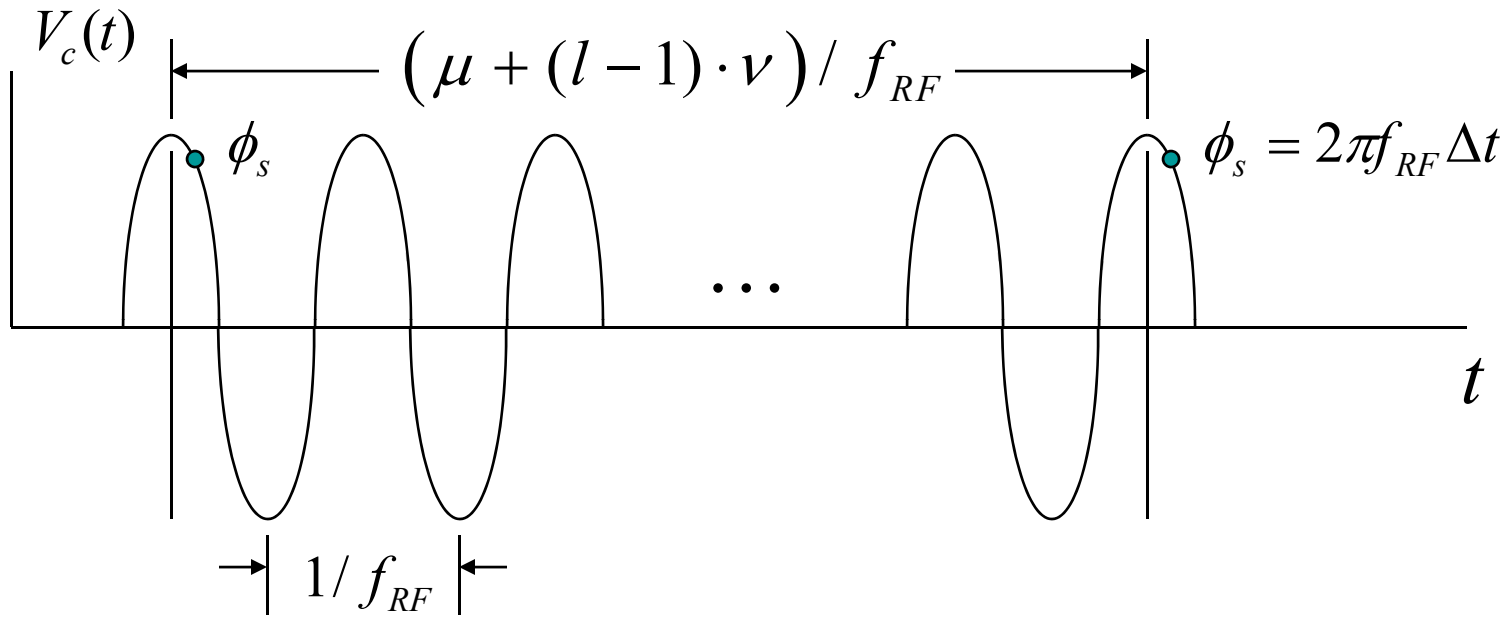
⊗ Magnetic Field

$\mu = 4$

$\nu = 2$

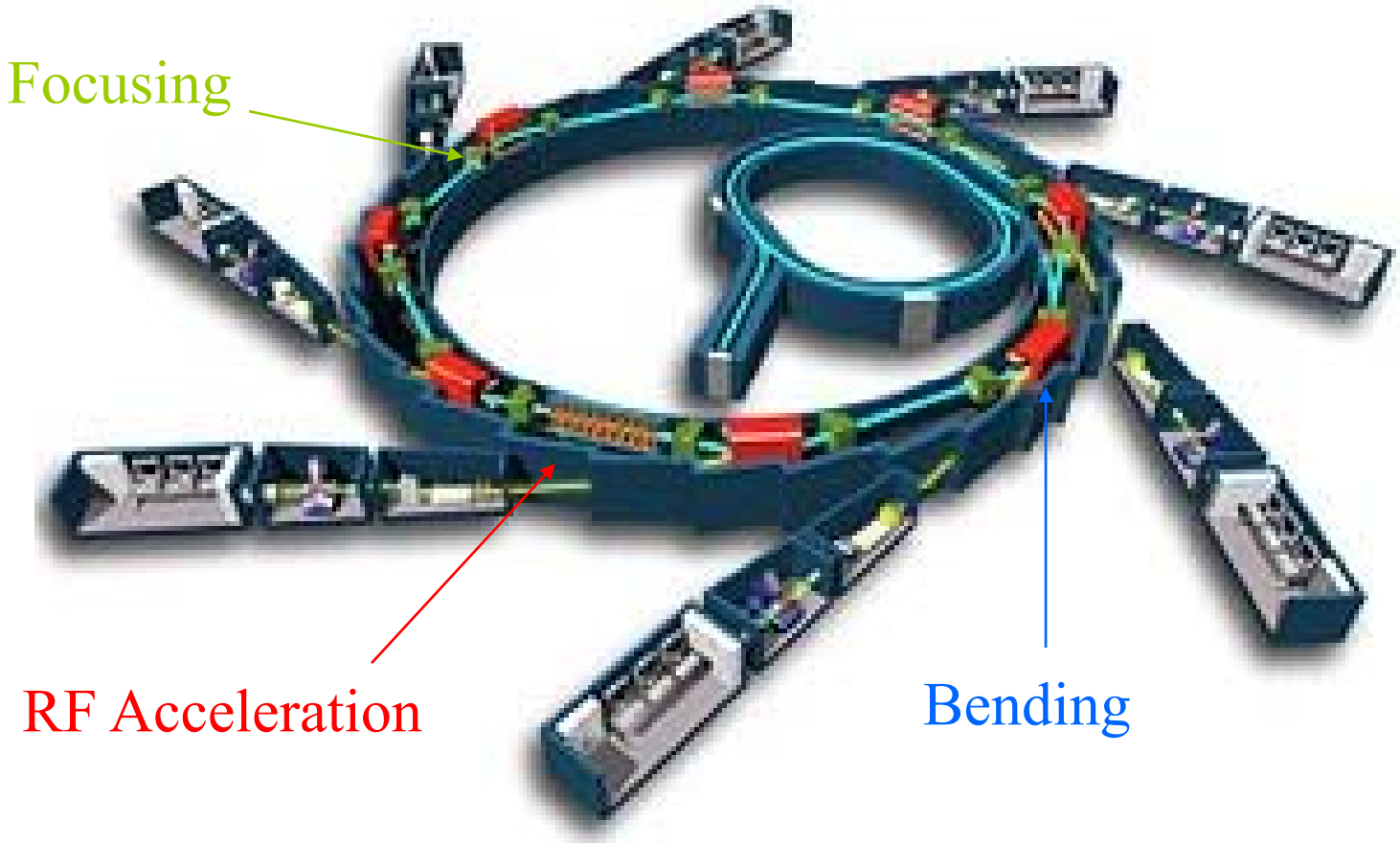
Phase Stability

Invented independently by Veksler (for microtrons!) and McMillan



Electrons arriving EARLY get more energy, have a longer path, and arrive later on the next pass. Extremely important discovery in accelerator physics. McMillan used same idea to design first electron synchrotron.

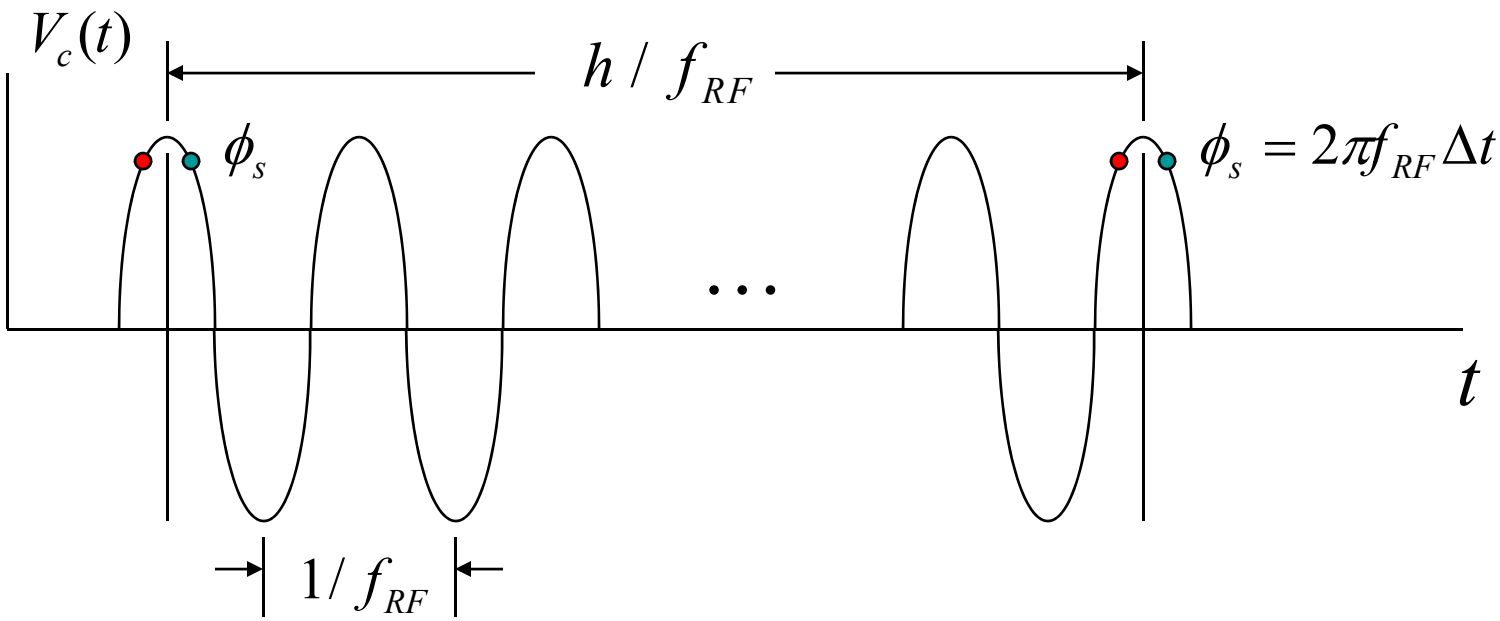
Generic Modern Synchrotron



Spokes are user stations for this X-ray ring source

Synchrotron Phase Stability

Edwin McMillan discovered phase stability independently of Veksler and used the idea to design first large electron synchrotron.



$$h = L f_{RF} / \beta c$$

Harmonic number: # of RF oscillations in a revolution

Transition Energy

Beam energy where speed increment effect balances path length change effect on accelerator revolution frequency. Revolution frequency independent of beam energy to linear order. We will calculate in a few weeks

- Below Transistion Energy: Particles arriving EARLY get less acceleration and speed increment, and arrive later, with repect to the center of the bunch, on the next pass. Applies to heavy particle synchrotrons during first part of acceleration when the beam is non-relativistic and accelerations still produce velocity changes.

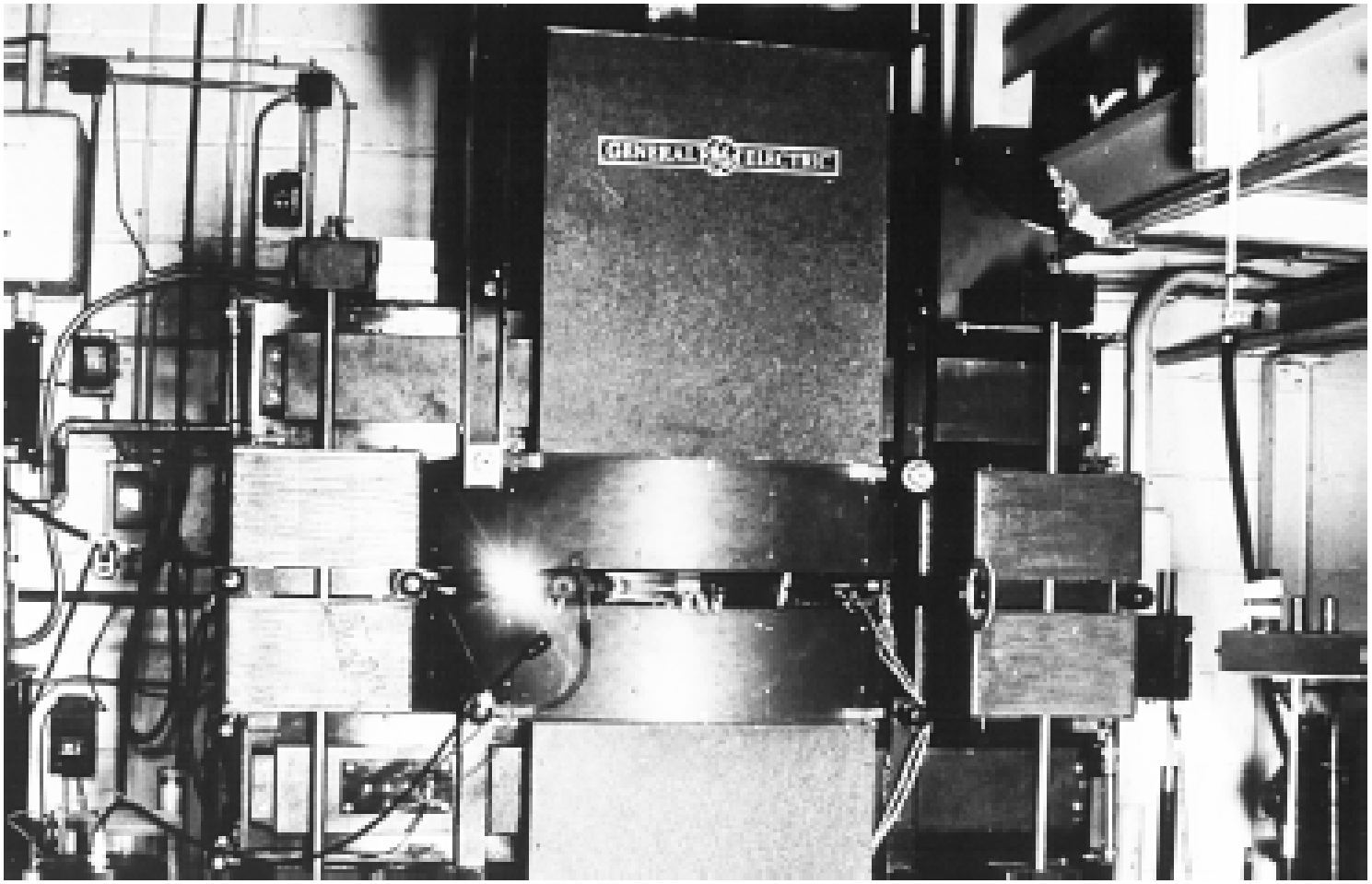
- Above Transistion Energy: Particles arriving EARLY get more energy, have a longer path, and arrive later on the next pass. Applies for electron synchrotrons and heavy particle synchrotrons when approach relativistic velocities. As seen before, Microtrons operate here.

Ed McMillan

Vacuum chamber for electron synchrotron being packed for shipment to Smithsonian

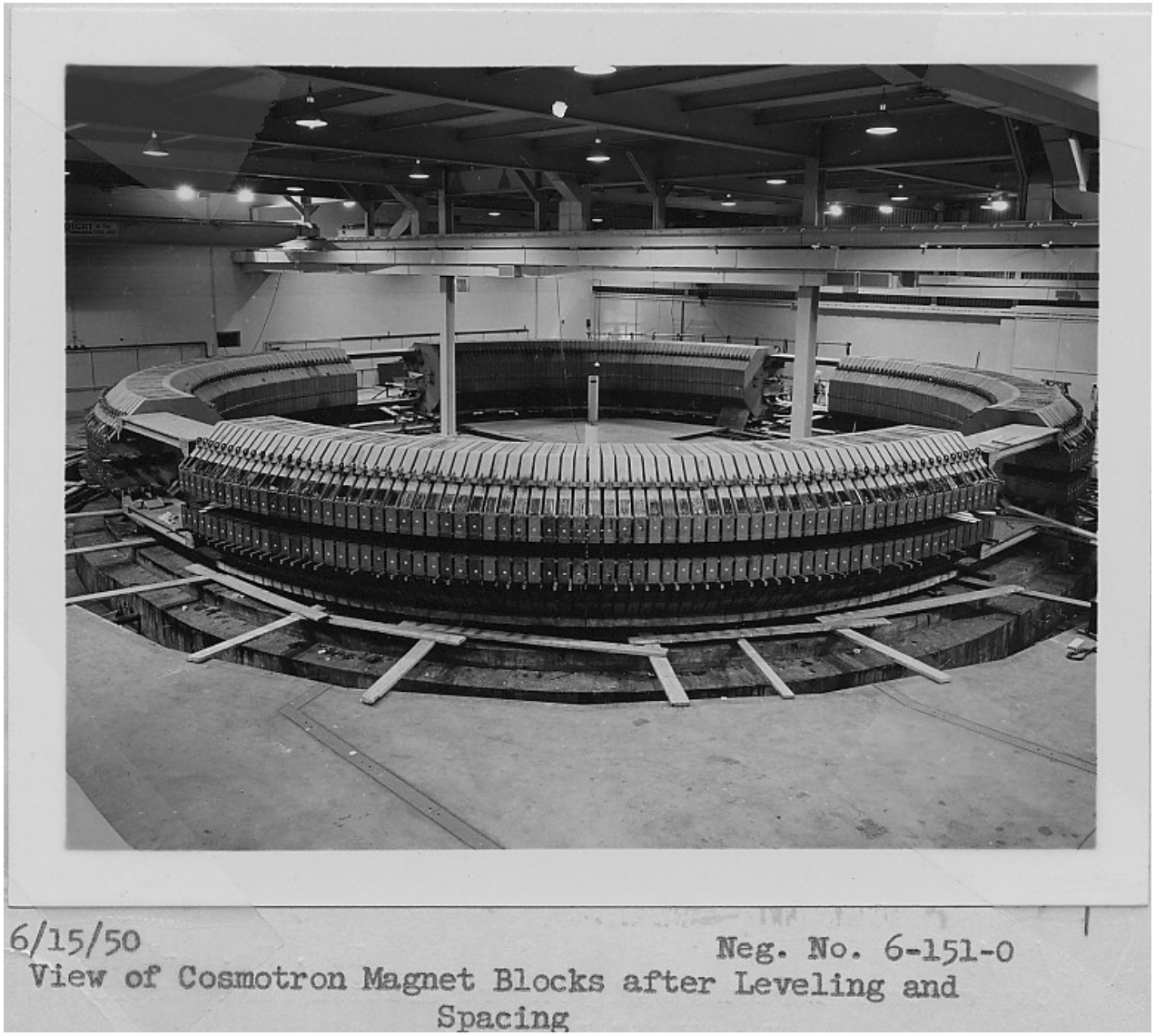
Full Electron Synchrotron

GE Electron Synchrotron

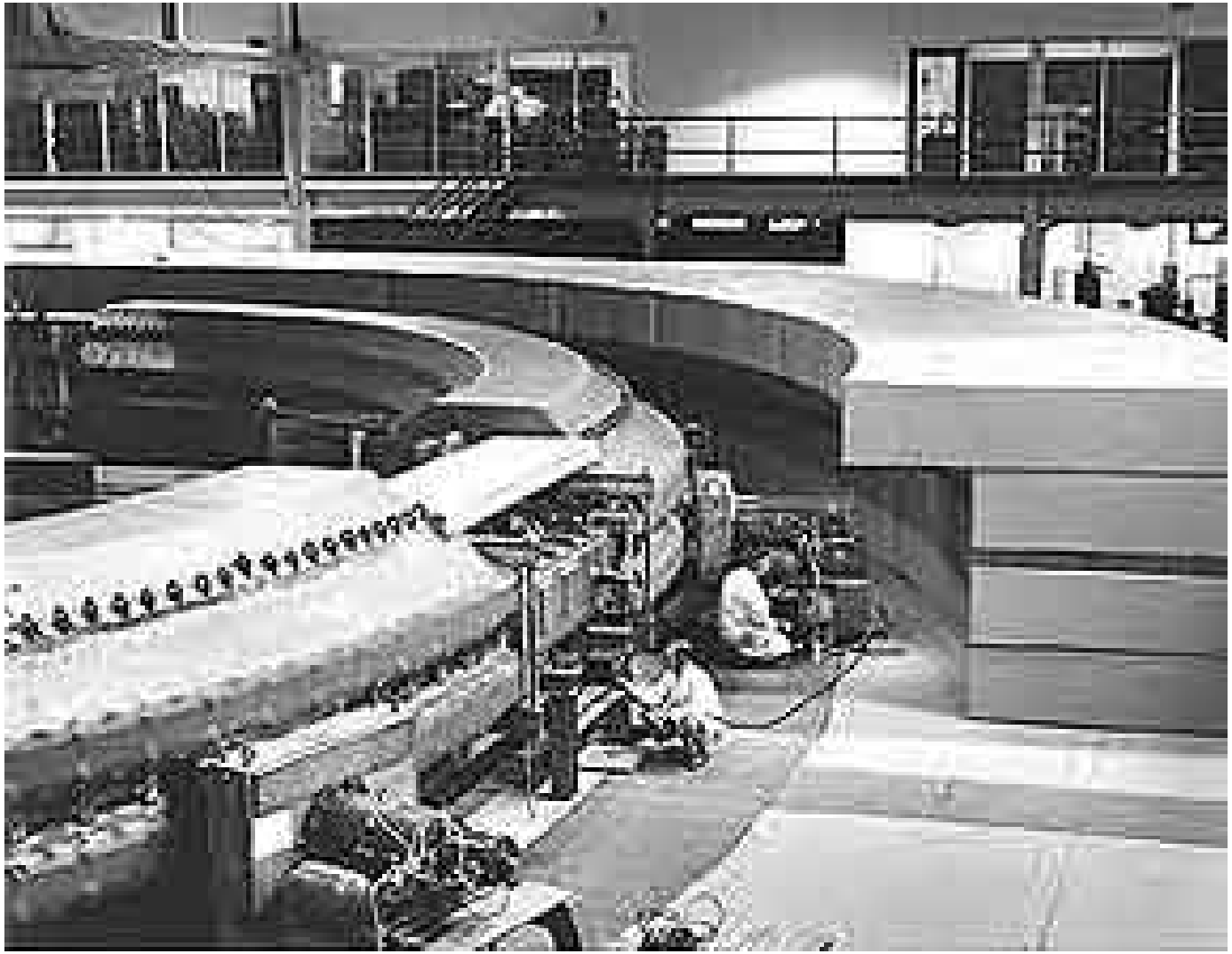


Elder, F. R.; Gurewitsch, A. M.; Langmuir, R. V.; Pollock, H. C., "Radiation from Electrons in a Synchrotron" (1947) *Physical Review*, vol. 71, Issue 11, pp. 829-830

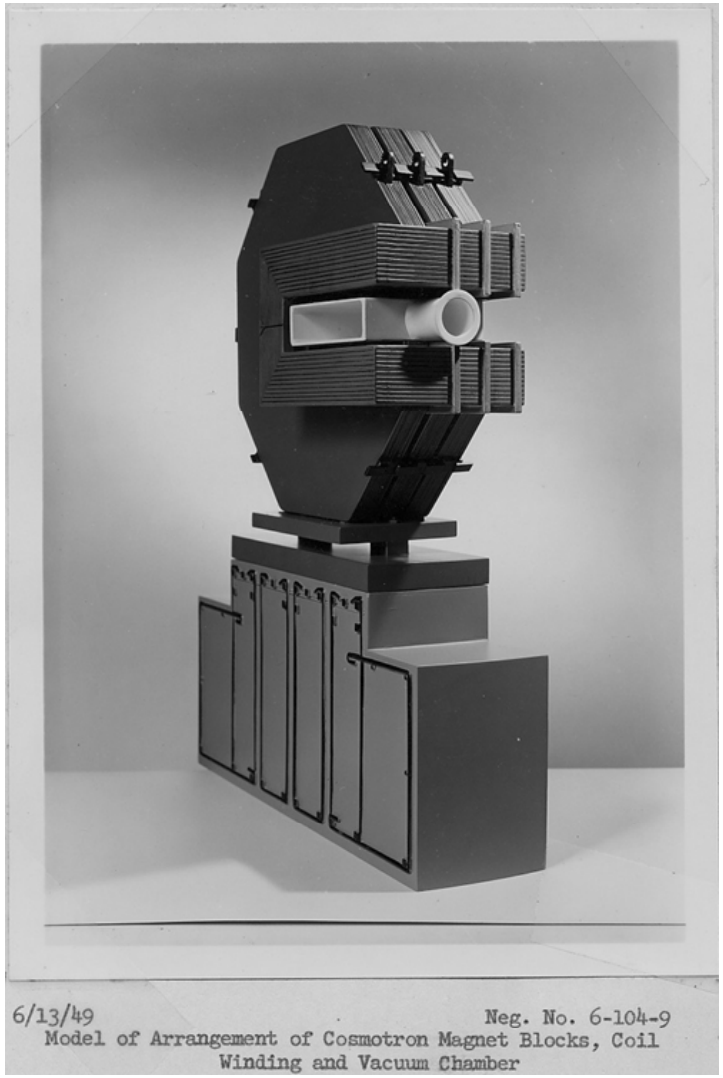
Cosmotron (First GeV Accelerator)

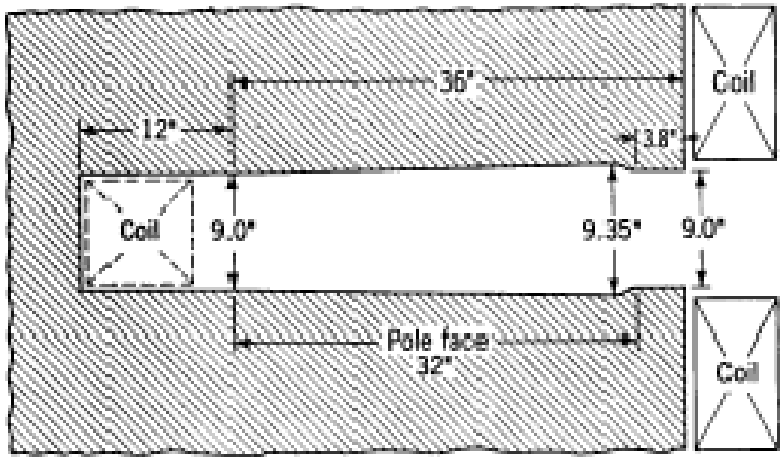


BNL Cosmotron and Shielding



Cosmotron Magnet





The Cosmotron magnet

Cosmotron People

E.Courant -Lattice Designer

Stan Livingston - Boss

Snyder -theorist

Christofilos - inventor

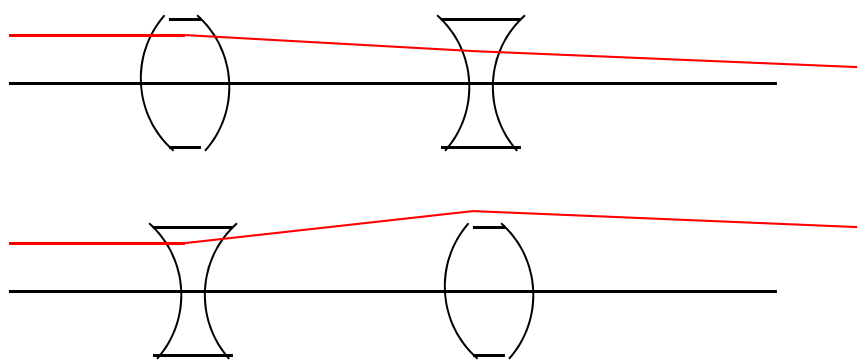
Bevatron

Designed to discover the antiproton; Largest Weak Focusing Synchrotron

Strong Focusing

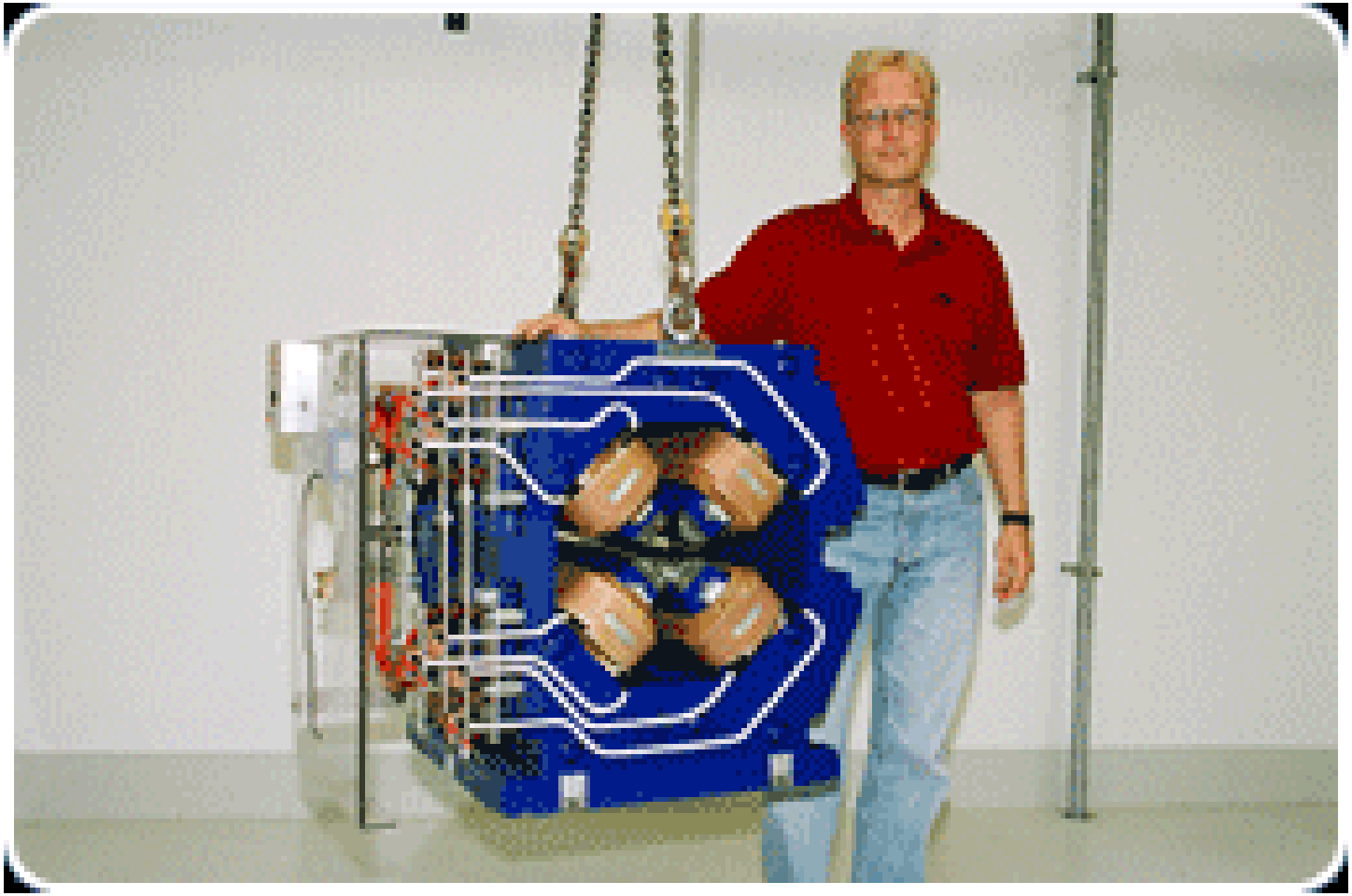
- Betatron oscillation work has showed us that, apart from bend plane focusing, a shaped field that focuses in one transverse direction, defocuses in the other
- Question: is it possible to develop a system that focuses in both directions simultaneously?

- Strong focusing: alternate the signs of focusing and defocusing: get net focusing!!



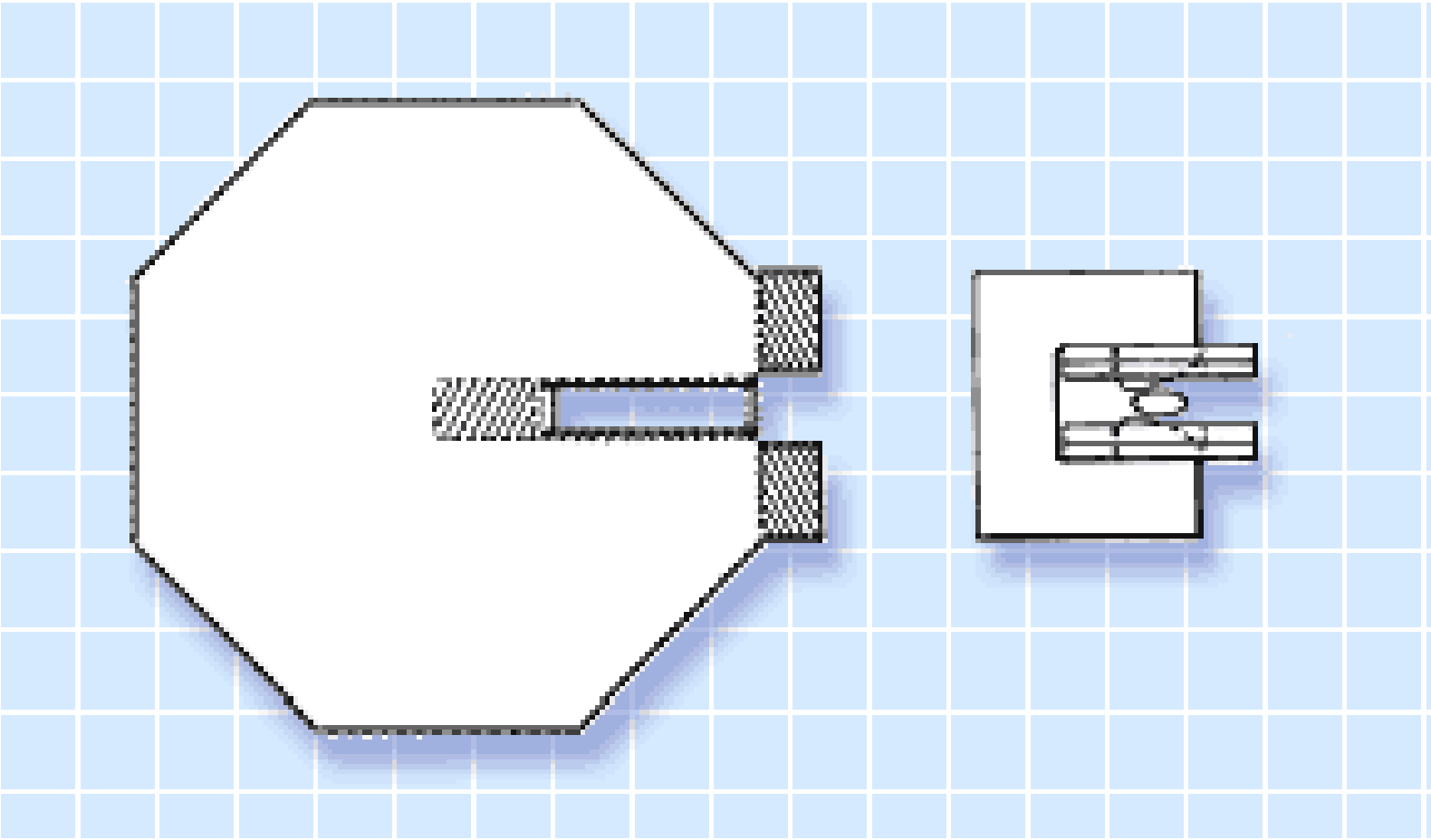
Order doesn't matter

Linear Magnetic Lenses: Quadrupoles



Source: Danfysik Web site

Weak vs. Strong Benders



Comment on Strong Focusing

Last time neglected to mention one main advantage of strong focusing. In weak focusing machines, $n < 1$ for stability. Therefore, the fall-off distance, or field gradient cannot be too high. There is no such limit for strong focusing.

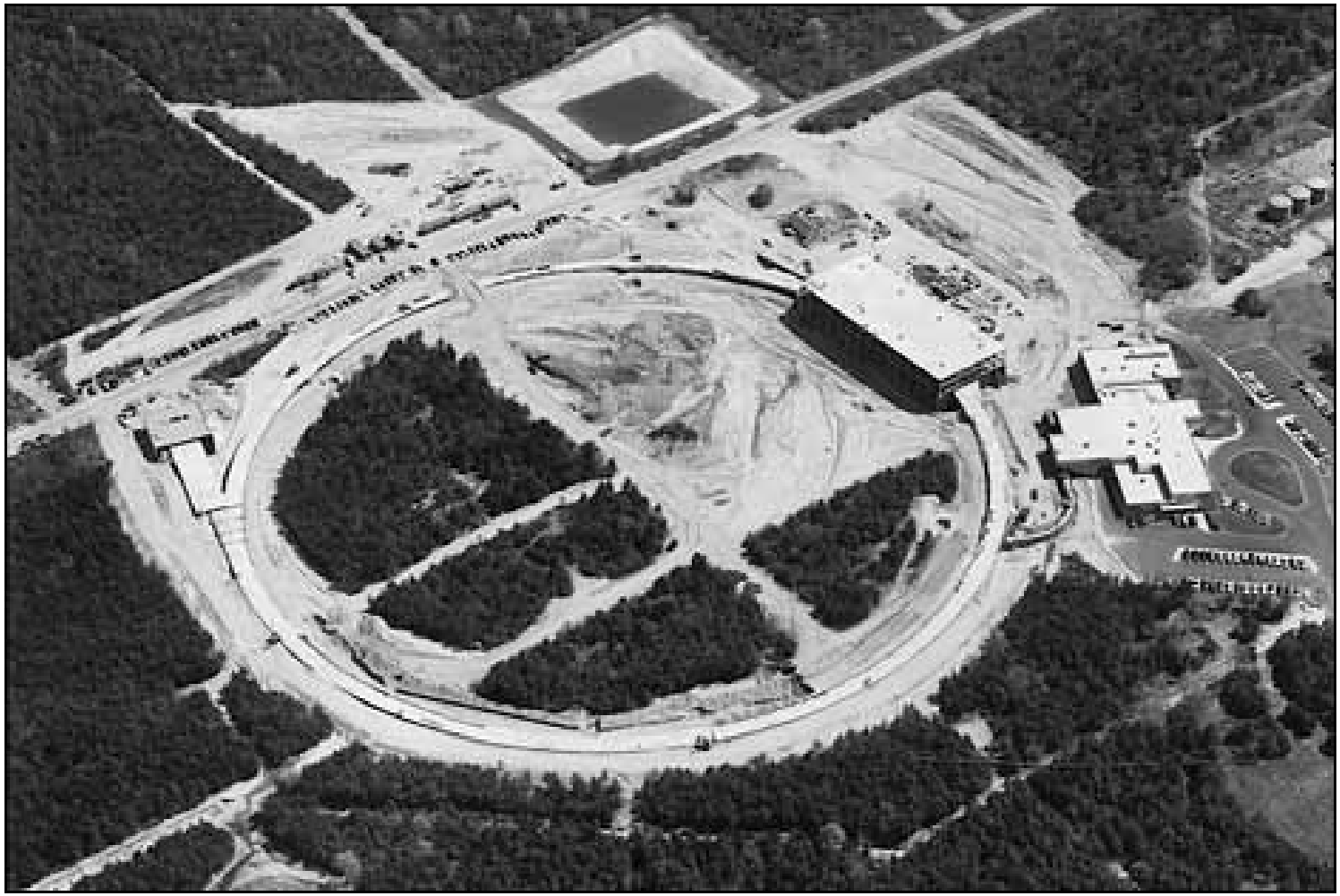
$$n \gg 1$$

is now allowed, leading to large field gradients and relatively short focal length magnetic lenses. This tighter focusing is what allows smaller beam sizes. Focusing gradients now limited only by magnet construction issues (pole magnetic field limits).

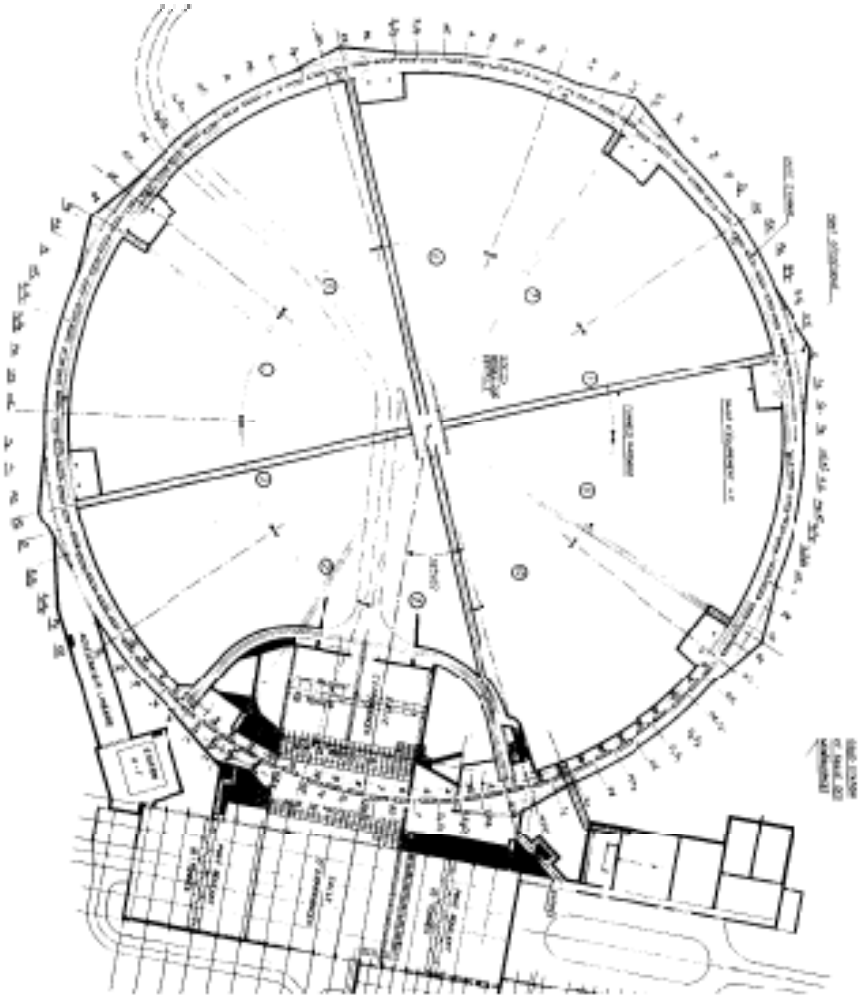
First Strong-Focusing Synchrotron

Cornell 1 GeV Electron Synchrotron (LEPP-AP Home Page)

Alternating Gradient Synchrotron (AGS)



CERN PS



25 GeV Proton Synchrotron

CERN SPS

Eventually 400 GeV protons and antiprotons

FNAL

First TeV-scale accelerator; Large Superconducting Benders

LEP Tunnel (Now LHC!)

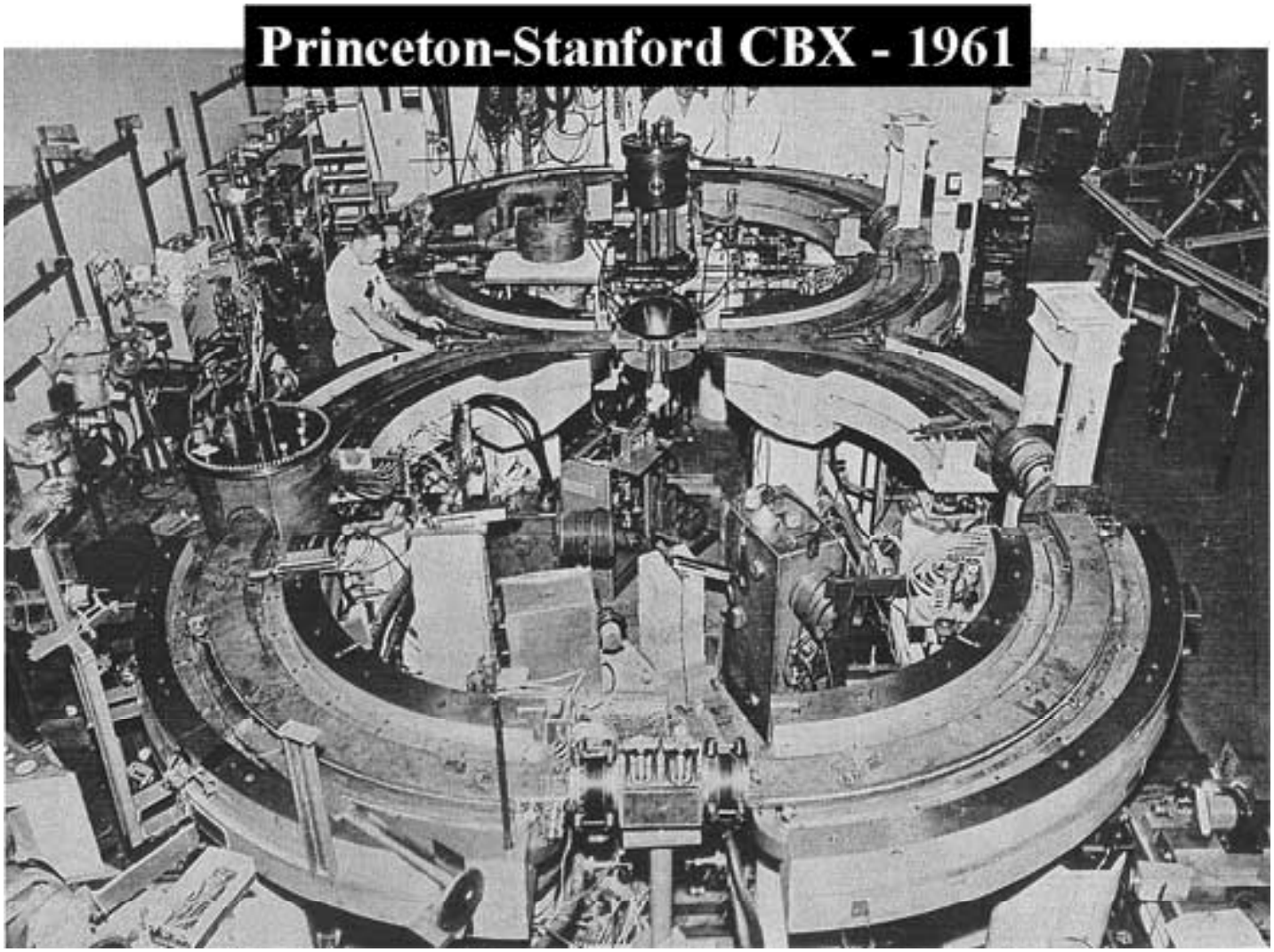
Empty

LHC

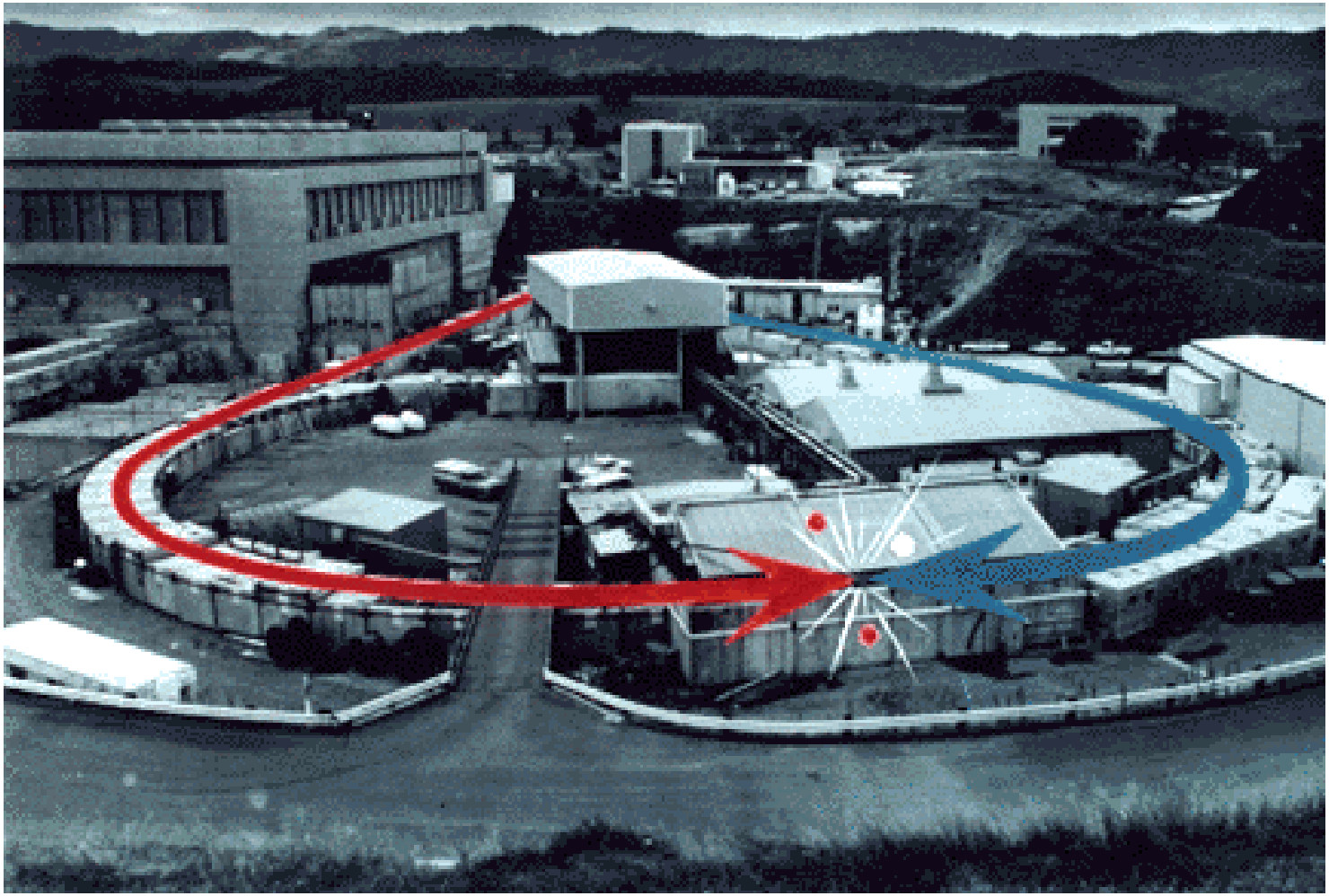
Storage Rings

- Some modern accelerators are designed not to "accelerate" much at all, but to "store" beams for long periods of time that can be usefully used by experimental users.
  - Colliders for High Energy Physics. Accelerated beam-accelerated beam collisions are much more energetic than accelerated beam-target collisions. To get to the highest beam energy for a given acceleration system design a collider
  - Electron storage rings for X-ray production: circulating electrons emit synchrotron radiation for a wide variety of experimental purposes.

Princeton-Stanford Collider



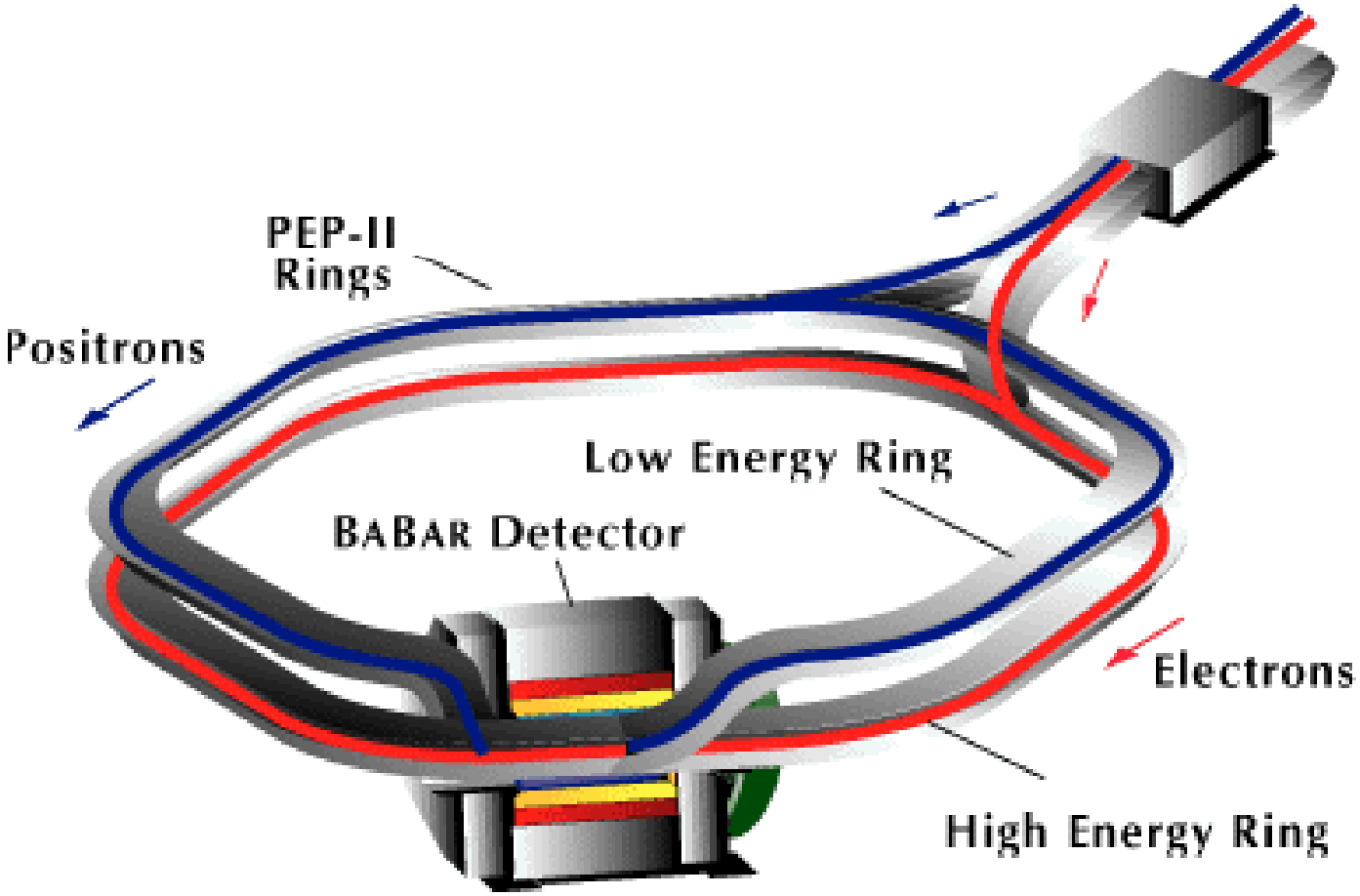
SPEAR



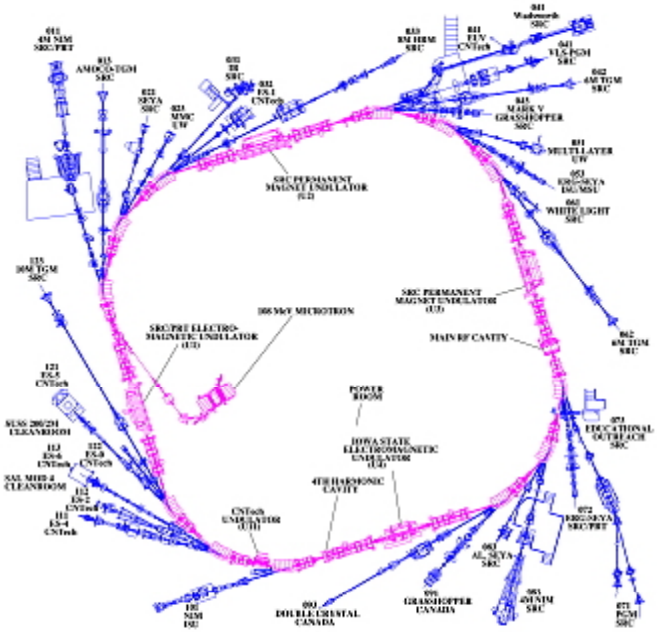
Eventually became leading synchrotron radiation machine

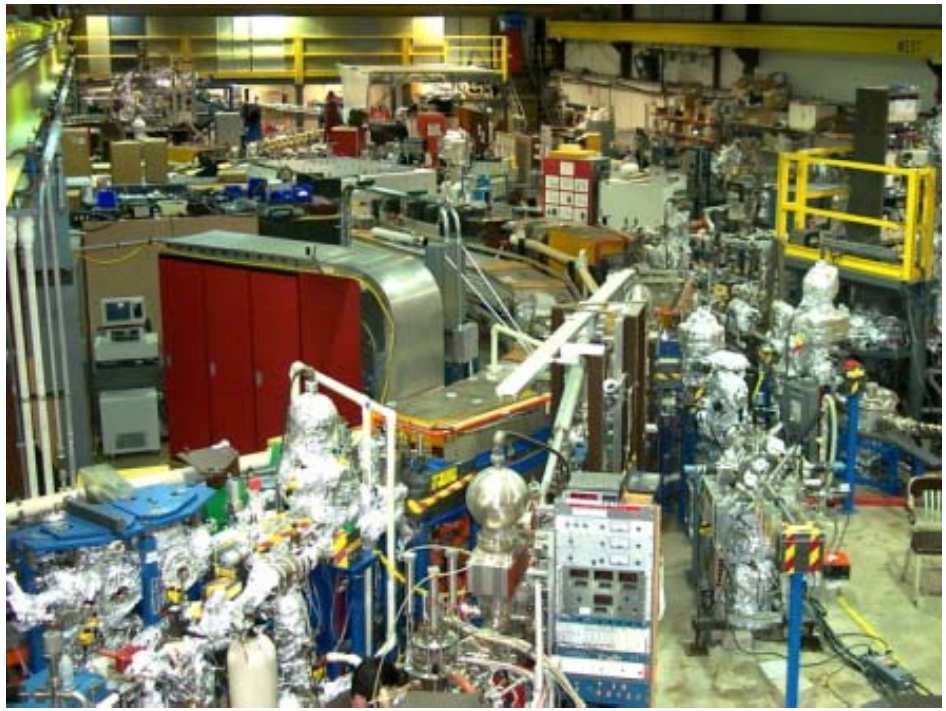
Cornell 10 GeV ES and CESR

SLAC's PEP II B-factory



ALADDIN at Univ. of Wisconsin

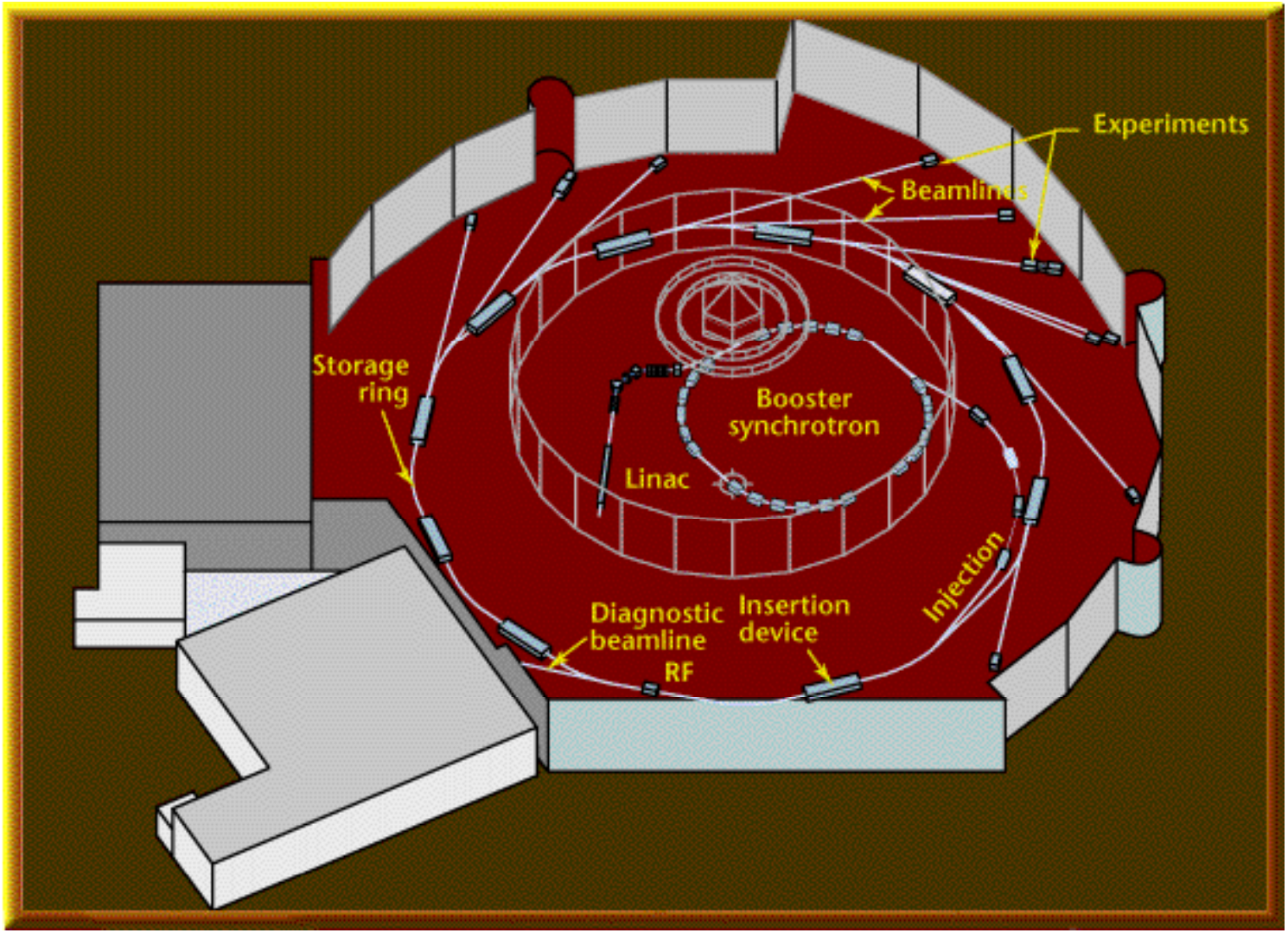




VUV Ring at NSLS

VUV ring “uncovered”

Berkeley's ALS



Argonne APS

ESRF

Comment on Strong Focusing

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$$n \gg 1$$

is now allowed, leading to large field gradients and relatively short focal length magnetic lenses. This tighter focusing is what allows smaller beam sizes. Focusing gradients now limited only by magnet construction issues (pole magnetic field limits).

Linear Beam Optics Outline

- Particle Motion in the Linear Approximation
- Some Geometry of Ellipses
- Ellipse Dimensions in the β-function Description
- Area Theorem for Linear Transformations
- Phase Advance for a Unimodular Matrix
  - Formula for Phase Advance
  - Matrix Twiss Representation
  - Invariant Ellipses Generated by a Unimodular Linear Transformation
- Detailed Solution of Hill's Equation
  - General Formula for Phase Advance
  - Transfer Matrix in Terms of β-function
  - Periodic Solutions
- Non-periodic Solutions
  - Formulas for β-function and Phase Advance
- Beam Matching

Linear Particle Motion

Fundamental Notion: The *Design Orbit* is a path in an Earth-fixed reference frame, i.e., a differentiable mapping from [0,1] to points within the frame. As we shall see as we go on, it generally consists of *arcs of circles* and *straight lines*.

$$\sigma : [0,1] \rightarrow \mathrm{R}^3$$

$$\sigma \quad \rightarrow \vec{X}(\sigma) = \left(X(\sigma), Y(\sigma), Z(\sigma) \right)$$

Fundamental Notion: *Path Length*

$$ds = \sqrt{\left(\frac{dX}{d\sigma}\right)^2 + \left(\frac{dY}{d\sigma}\right)^2 + \left(\frac{dZ}{d\sigma}\right)^2} \, d\sigma$$

The *Design Trajectory* is the path specified in terms of the path length in the Earth-fixed reference frame. For a relativistic accelerator where the particles move at the velocity of light, $L_{tot}=ct_{tot}$.

$$s:[0,L_{tot}]\rightarrow \mathrm{R}^3$$

$$s \;\rightarrow\; \vec{X}(s)=\left(X(s),Y(s),Z(s)\right)$$

The first step in designing any accelerator, is to specify bending magnet locations that are consistent with the arc portions of the Design Trajectory.

Betatron Design Trajectory

$$s : [0, 2\pi R] \to \mathrm{R}^3$$

$$s \;\to\; \vec{X}(s) = \left(R\cos(s/R), R\sin(s/R), 0\right)$$

Use path length s as independent variable instead of t in the dynamical equations.

$$\frac{d}{ds} = \frac{1}{\Omega_c R}\frac{d}{dt}$$

Betatron Motion in *s*

$$\frac{d^2\delta r}{dt^2} + \left(1-n\right)\Omega_c^2\delta r = \Omega_c^2 R\frac{\Delta p}{p}$$

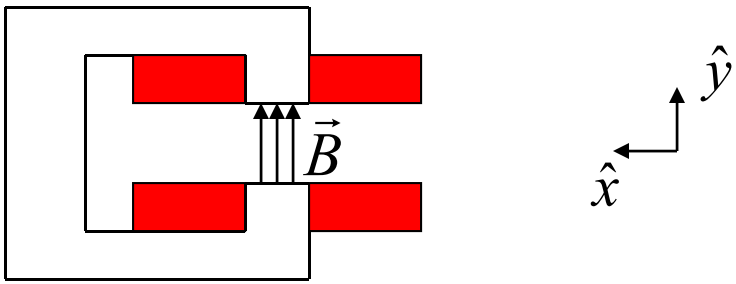
$$\frac{d^2\delta z}{dt^2} + n\Omega_c^2\delta z = 0$$

$$\Downarrow$$

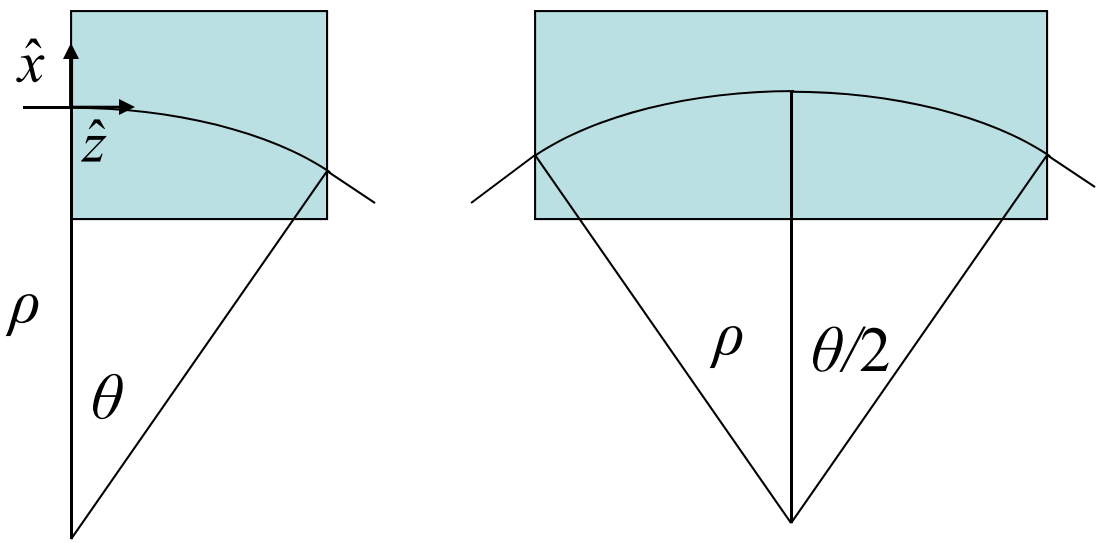
$$\frac{d^2\delta r}{ds^2} + \frac{\left(1-n\right)}{R^2}\delta r = \frac{1}{R}\frac{\Delta p}{p}$$

$$\frac{d^2\delta z}{ds^2} + \frac{n}{R^2}\delta z = 0$$

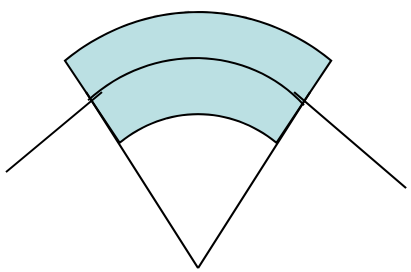
Bend Magnet Geometry



Rectangular Magnet of Length L



Sector Magnet



Bend Magnet Trajectory

For a uniform magnetic field

$$\frac{d(\gamma m \vec{V})}{dt} = \left[\vec{E} + \vec{V} \times \vec{B} \right]$$

$$\frac{d(\gamma m V_x)}{dt} = -q V_z B_y$$

$$\frac{d(\gamma m V_z)}{dt} = q V_x B_y$$

$$\frac{d^2 V_x}{dt^2} + \Omega_c^2 V_x = 0 \qquad \frac{d^2 V_z}{dt^2} + \Omega_c^2 V_z = 0$$

For the solution satisfying boundary conditions: $\vec{X}(0) = 0 \qquad \vec{V}(0) = V_{0z}\hat{z}$

$$X(t) = \frac{p}{qB_y}\left(\cos(\Omega_c t) - 1\right) = \rho\left(\cos(\Omega_c t) - 1\right) \qquad \Omega_c = qB_y / \gamma m$$

$$Z(t) = \frac{p}{qB_y}\sin(\Omega_c t) = \rho \sin(\Omega_c t)$$

Magnetic Rigidity

The magnetic rigidity is:

$$B\rho = \left|B_y \rho\right| = \frac{p}{|q|}$$

It depends only on the particle momentum and charge, and is a convenient way to characterize the magnetic field. Given magnetic rigidity and the required bend radius, the required bend field is a simple ratio. Note particles of momentum 100 MeV/*c* have a rigidity of 0.334 T m.

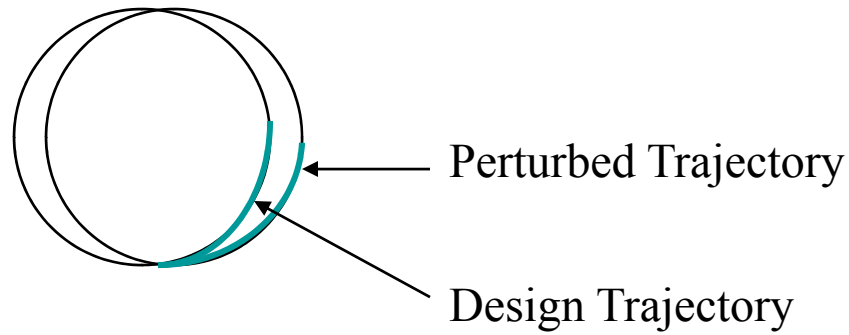
Long Dipole Magnet

$$BL = B\rho\left(2\sin\left(\theta/2\right)\right)$$

Normal Incidence (or exit) Dipole Magnet

$$BL = B\rho\sin\left(\theta\right)$$

Natural Focusing in Bend Plane



Can show that for either a displacement perturbation or angular perturbation from the design trajectory

$$\frac{d^2x}{ds^2} = -\frac{x}{\rho_x^2(s)} \qquad \frac{d^2y}{ds^2} = -\frac{y}{\rho_y^2(s)}$$

Quadrupole Focusing

$$\vec{B}(x,y) = B'(s)(x\hat{y} + y\hat{x})$$

$$\gamma m \frac{d\mathrm{v}_\mathrm{x}}{ds} = -qB'(s)x \qquad \gamma m \frac{d\mathrm{v}_\mathrm{y}}{ds} = qB'(s)y$$

$$\frac{d^2x}{ds^2} + \frac{B'(s)}{B\rho}x = 0 \qquad \frac{d^2y}{ds^2} - \frac{B'(s)}{B\rho}y = 0$$

Combining with the previous slide

$$\frac{d^2x}{ds^2} + \left[\frac{1}{\rho_x^2(s)} + \frac{B'(s)}{B\rho}\right]x = 0 \qquad \frac{d^2y}{ds^2} + \left[\frac{1}{\rho_y^2(s)} - \frac{B'(s)}{B\rho}\right]y = 0$$

Hill's Equation

Define focusing strengths (with units of m^{-2})

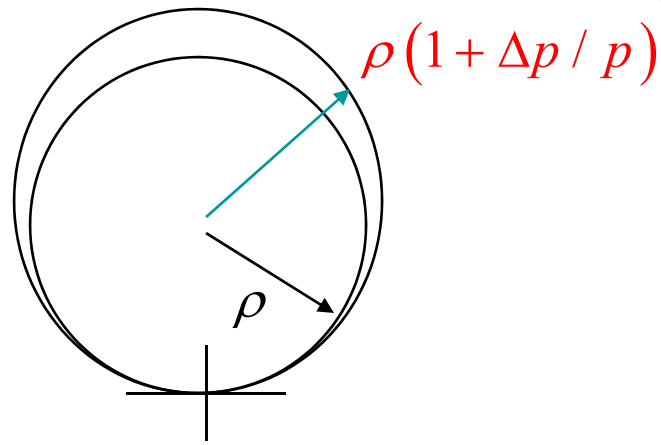
$$k_x(s) = \frac{1}{\rho_x^2(s)} + \frac{B'(s)}{B\rho} \qquad k_y = \frac{1}{\rho_y^2(s)} - \frac{B'(s)}{B\rho}$$

$$\frac{d^2x}{ds^2} + k_x(s)x = 0 \qquad \frac{d^2y}{ds^2} + k_y(s)y = 0$$

Note that this is like the harmonic oscillator, or exponential for constant *K,* but more general in that the focusing strength, and hence oscillation frequency depends on *s*

Energy Effects

$$\Delta x(s) = \frac{p}{eB_y}\frac{\Delta p}{p}\left(1-\cos\left(s/\rho\right)\right)$$



This solution is not a solution to Hill's equation directly, but *is* a solution to the inhomogeneous Hill's Equations

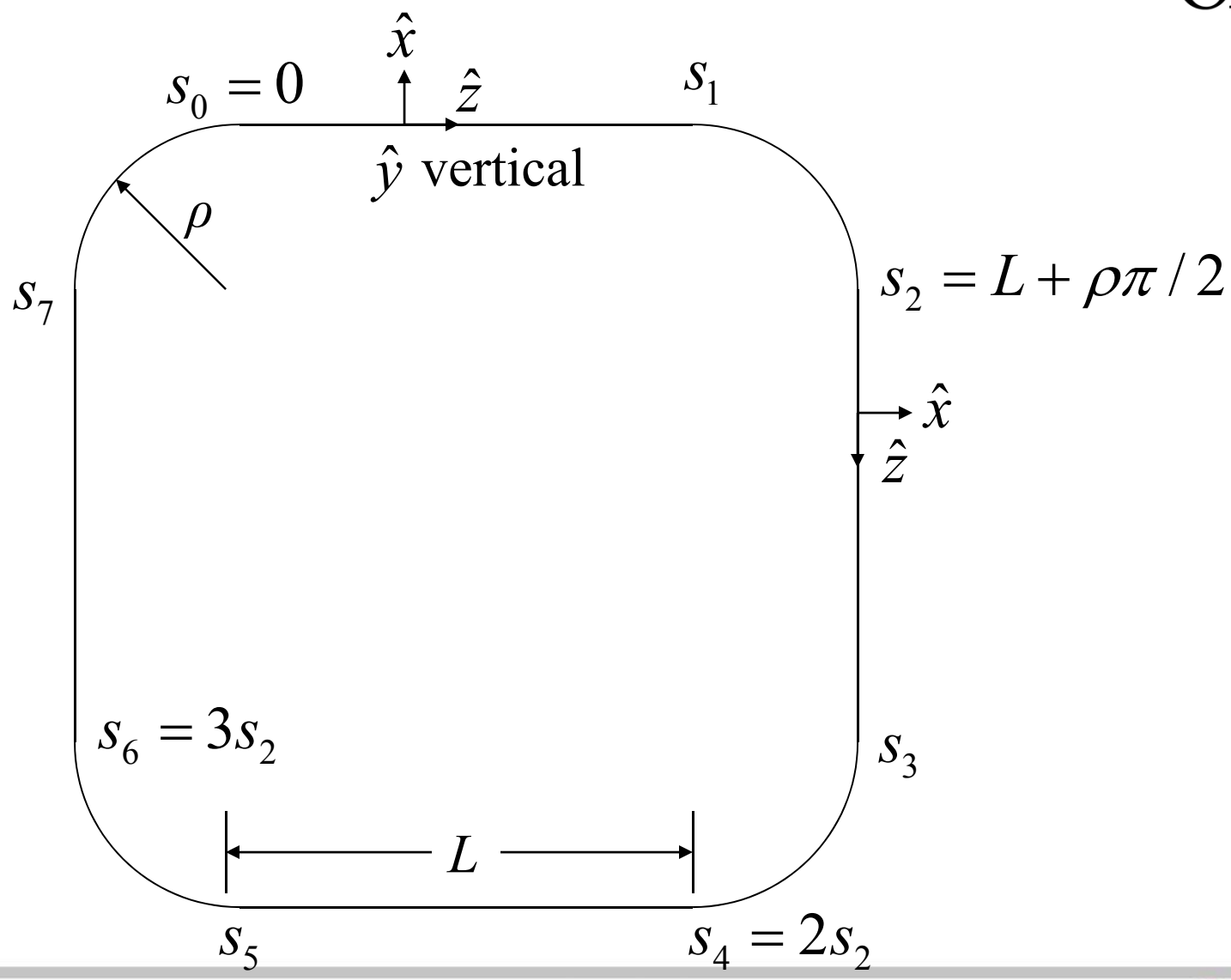
$$\frac{d^2x}{ds^2}+\left[\frac{1}{\rho_x^2(s)}+\frac{B'(s)}{B\rho}\right]x=\frac{1}{\rho_x(s)}\frac{\Delta p}{p}$$

$$\frac{d^2y}{ds^2}+\left[\frac{1}{\rho_y^2(s)}-\frac{B'(s)}{B\rho}\right]y=\frac{1}{\rho_y(s)}\frac{\Delta p}{p}$$

Comment on Design Trajectory

The notion of specifying curves in terms of their path length is standard in courses on the vector analysis of curves. A good discussion in a Calculus book is Thomas, *Calculus and Analytic Geometry*, 4th Edition, Articles 14.3-14.5. Most vector analysis books have a similar, and more advanced discussion under the subject of "Frenet-Serret Equations". Because all of our design trajectories involve only arcs of circles and straight lines (dipole magnets and the drift regions between them define the orbit), we can concentrate on a simplified set of equations that "only" involve the radius of curvature of the design orbit. It may be worthwhile giving a simple example.

4-Fold Symmetric Synchrotron



Its Design Trajectory

$$
\begin{aligned}
&(0,0,s) && 0 < s < L = s_1 \\
&(0,0,L) + \rho\left(\cos\left((s-s_1)/\rho\right) - 1, 0, \sin\left((s-s_1)/\rho\right)\right) && s_1 < s < s_2 \\
&(-\rho, 0, L+\rho) + (s-s_2)(-1,0,0) && s_2 < s < s_3 \\
&(-L-\rho, 0, L+\rho) + \rho\left(-\sin\left((s-s_3)/\rho\right), 0, \cos\left((s-s_3)/\rho\right) - 1\right) && s_3 < s < s_4 \\
&(-L-2\rho, 0, L) + (s-s_4)(0,0,-1) && s_4 < s < s_5 \\
&(-L-2\rho, 0, 0) + \rho\left(1 - \cos\left((s-s_5)/\rho\right), 0, -\sin\left((s-s_5)/\rho\right)\right) && s_5 < s < s_6 \\
&(-L-\rho, 0, -\rho) + (s-s_6)(1,0,0) && s_6 < s < s_7 \\
&(-\rho, 0, -\rho) + \rho\left(\sin\left((s-s_7)/\rho\right), 0, 1 - \cos\left((s-s_7)/\rho\right)\right) && s_7 < s < 4s_2
\end{aligned}
$$

Inhomogeneous Hill's Equations

Fundamental transverse equations of motion in particle accelerators for small deviations from design trajectory

$$\frac{d^2x}{ds^2}+\left[\frac{1}{\rho_x^2(s)}+\frac{B'(s)}{B\rho}\right]x=\frac{1}{\rho_x(s)}\frac{\Delta p}{p}$$

$$\frac{d^2y}{ds^2}+\left[\frac{1}{\rho_y^2(s)}-\frac{B'(s)}{B\rho}\right]y=\frac{1}{\rho_y(s)}\frac{\Delta p}{p}$$

ρ radius of curvature for bends, B' transverse field gradient for magnets that focus (positive corresponds to horizontal focusing), $\Delta p/p$ momentum deviation from design momentum. Homogeneous equation is 2nd order *linear* ordinary differential equation.

Dispersion

From theory of linear ordinary differential equations, the general solution to the inhomogeneous equation is the sum of **any** solution to the inhomogeneous equation, called the particular integral, plus two linearly independent solutions to the homogeneous equation, whose amplitudes may be adjusted to account for boundary conditions on the problem.

$$x(s)=x_p(s)+A_x x_1(s)+B_x x_2(s) \qquad y(s)=y_p(s)+A_y y_1(s)+B_y y_2(s)$$

Because the inhomogeneous terms are proportional to *Δp/p*, the particular solution can generally be written as

$$x_p(s)=D_x(s)\frac{\Delta p}{p} \qquad y_p(s)=D_y(s)\frac{\Delta p}{p}$$

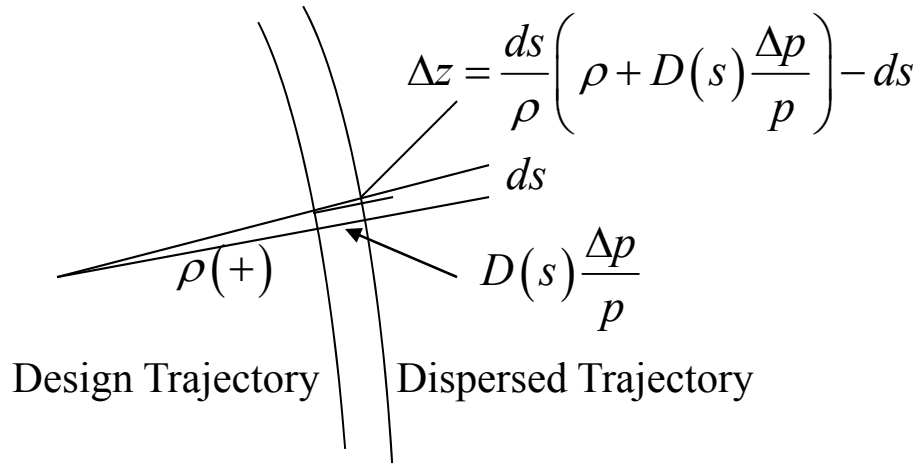
where the dispersion functions satisfy

$$\frac{d^2 D_x}{ds^2}+\left[\frac{1}{\rho_x^2(s)}+\frac{B'(s)}{B\rho}\right]D_x=\frac{1}{\rho_x(s)} \qquad \frac{d^2 D_y}{ds^2}+\left[\frac{1}{\rho_y^2(s)}-\frac{B'(s)}{B\rho}\right]D_y=\frac{1}{\rho_y(s)}$$

M_{56}

In addition to the transverse effects of the dispersion, there are important effects of the dispersion along the direction of motion. The primary effect is to change the time-of-arrival of the off-momentum particle compared to the on-momentum particle which traverses the design trajectory.

$$d(\Delta z) = D(s)\frac{\Delta p}{p}\frac{ds}{\rho(s)}$$



$$M_{56} = \int_{s_1}^{s_2} \left\{ \frac{D_x(s)}{\rho_x(s)} + \frac{D_y(s)}{\rho_y(s)} \right\} ds$$

Solutions Homogeneous Eqn.

Dipole

$$\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} \cos\left((s-s_i)/\rho\right) & \rho\sin\left((s-s_i)/\rho\right) \\ -\sin\left((s-s_i)/\rho\right)/\rho & \cos\left((s-s_i)/\rho\right) \end{pmatrix} \begin{pmatrix} x(s_i) \\ \frac{dx}{ds}(s_i) \end{pmatrix}$$

Drift

$$\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} 1 & s-s_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(s_i) \\ \frac{dx}{ds}(s_i) \end{pmatrix}$$

Quadrupole in the focusing direction $k = B'/B\rho$

$$\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} \cos\left(\sqrt{k}\,(s-s_i)\right) & \sin\left(\sqrt{k}\,(s-s_i)\right)/\sqrt{k} \\ -\sqrt{k}\sin\left(\sqrt{k}\,(s-s_i)\right) & \cos\left(\sqrt{k}\,(s-s_i)\right) \end{pmatrix} \begin{pmatrix} x(s_i) \\ \frac{dx}{ds}(s_i) \end{pmatrix}$$

Thin Focusing Lens (limiting case when argument goes to zero!)

$$\begin{pmatrix} x(s+\varepsilon) \\ \frac{dx}{ds}(s+\varepsilon) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} x(s-\varepsilon) \\ \frac{dx}{ds}(s-\varepsilon) \end{pmatrix}$$

Thin Defocusing Lens: change sign of f

Solutions Homogeneous Eqn.

Dipole

$$\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} \cos\left((s-s_i)/\rho\right) & \rho\sin\left((s-s_i)/\rho\right) \\ -\sin\left((s-s_i)/\rho\right)/\rho & \cos\left((s-s_i)/\rho\right) \end{pmatrix} \begin{pmatrix} x(s_i) \\ \frac{dx}{ds}(s_i) \end{pmatrix}$$

Drift

$$\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} 1 & s-s_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(s_i) \\ \frac{dx}{ds}(s_i) \end{pmatrix}$$

Quadrupole in the focusing direction $k = B'/B\rho$

$$\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} \cos\left(\sqrt{k}\,(s-s_i)\right) & \sin\left(\sqrt{k}\,(s-s_i)\right)/\sqrt{k} \\ -\sqrt{k}\,\sin\left(\sqrt{k}\,(s-s_i)\right) & \cos\left(\sqrt{k}\,(s-s_i)\right) \end{pmatrix} \begin{pmatrix} x(s_i) \\ \frac{dx}{ds}(s_i) \end{pmatrix}$$

Quadrupole in the defocusing direction $k = B'/B\rho$

$$\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} \cosh\left(\sqrt{-k}\,(s-s_i)\right) & \sinh\left(\sqrt{-k}\,(s-s_i)\right)/\sqrt{-k} \\ \sqrt{-k}\,\sinh\left(\sqrt{-k}\,(s-s_i)\right) & \cosh\left(\sqrt{-k}\,(s-s_i)\right) \end{pmatrix} \begin{pmatrix} x(s_i) \\ \frac{dx}{ds}(s_i) \end{pmatrix}$$

Transfer Matrices

Dipole with bend Θ (put coordinate of final position in solution)

$$\begin{pmatrix} x\left(s_{after}\right) \\ \frac{dx}{ds}\left(s_{after}\right) \end{pmatrix} = \begin{pmatrix} \cos\left(\Theta\right) & \rho\sin\left(\Theta\right) \\ -\sin\left(\Theta\right)/\rho & \cos\left(\Theta\right) \end{pmatrix} \begin{pmatrix} x\left(s_{before}\right) \\ \frac{dx}{ds}\left(s_{before}\right) \end{pmatrix}$$

Drift

$$\begin{pmatrix} x\left(s_{after}\right) \\ \frac{dx}{ds}\left(s_{after}\right) \end{pmatrix} = \begin{pmatrix} 1 & L_{drift} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x\left(s_{before}\right) \\ \frac{dx}{ds}\left(s_{before}\right) \end{pmatrix}$$

Quadrupole in the focusing direction length L

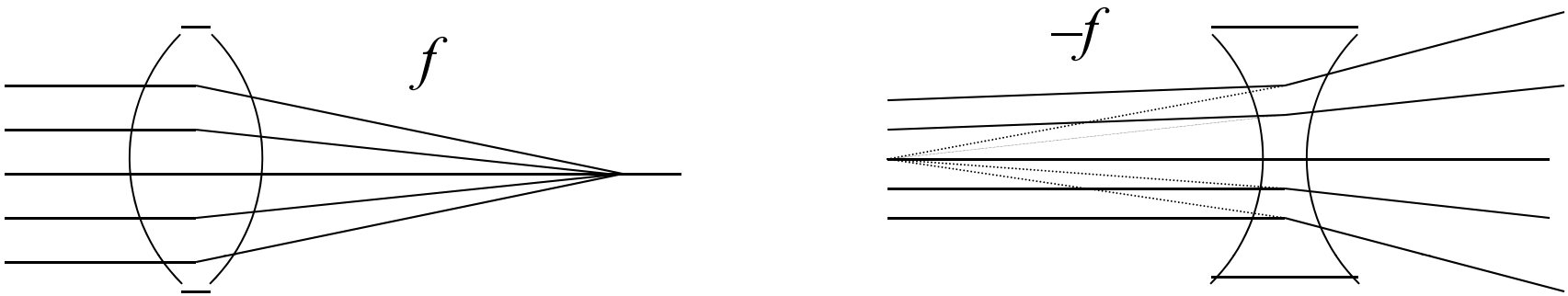
$$\begin{pmatrix} x\left(s_{after}\right) \\ \frac{dx}{ds}\left(s_{after}\right) \end{pmatrix} = \begin{pmatrix} \cos\left(\sqrt{k}L\right) & \sin\left(\sqrt{k}L\right)/\sqrt{k} \\ -\sqrt{k}\sin\left(\sqrt{k}L\right) & \cos\left(\sqrt{k}L\right) \end{pmatrix} \begin{pmatrix} x\left(s_{before}\right) \\ \frac{dx}{ds}\left(s_{before}\right) \end{pmatrix}$$

Quadrupole in the defocusing direction length L

$$\begin{pmatrix} x\left(s_{after}\right) \\ \frac{dx}{ds}\left(s_{after}\right) \end{pmatrix} = \begin{pmatrix} \cosh\left(\sqrt{-k}L\right) & \sinh\left(\sqrt{-k}L\right)/\sqrt{-k} \\ \sqrt{-k}\sinh\left(\sqrt{-k}L\right) & \cos\left(\sqrt{-k}L\right) \end{pmatrix} \begin{pmatrix} x\left(s_{before}\right) \\ \frac{dx}{ds}\left(s_{before}\right) \end{pmatrix}$$

Wille: pg. 71

Thin Lenses

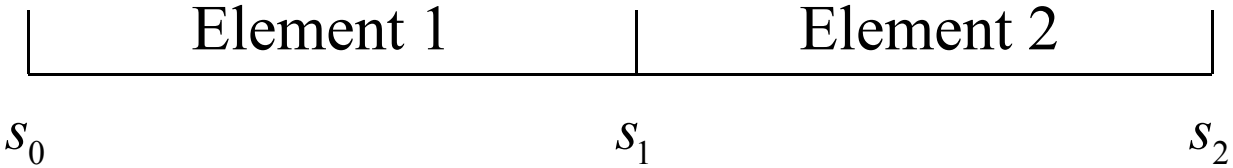


Thin Focusing Lens (limiting case when argument goes to zero!)

$$\begin{pmatrix} x\left(s_{lens}+\varepsilon\right) \\ \frac{dx}{ds}\left(s_{lens}+\varepsilon\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} x\left(s_{lens}-\varepsilon\right) \\ \frac{dx}{ds}\left(s_{lens}-\varepsilon\right) \end{pmatrix}$$

Thin Defocusing Lens: change sign of f

Composition Rule: Matrix Multiplication!



$$\begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix} = M_1 \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix} \qquad \begin{pmatrix} x(s_2) \\ x'(s_2) \end{pmatrix} = M_2 \begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix}$$

$$\begin{pmatrix} x(s_2) \\ x'(s_2) \end{pmatrix} = M_2 M_1 \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix}$$

More generally

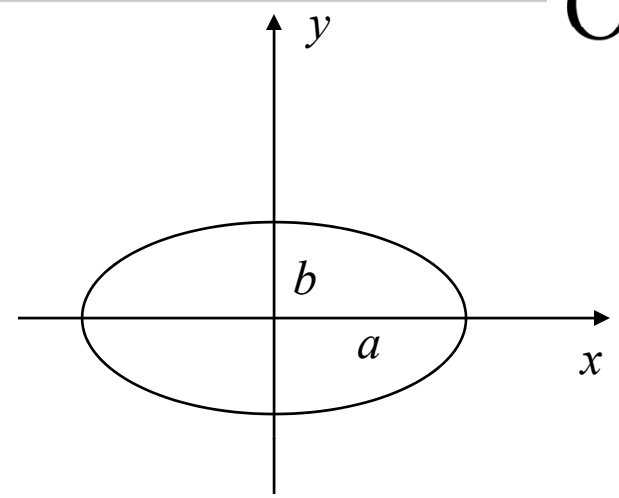
$$M_{tot} = M_N M_{N-1} \ldots M_2 M_1$$

Remember: First element farthest RIGHT

Some Geometry of Ellipses

Equation for an upright ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



In beam optics, the equations for ellipses are normalized (by multiplication of the ellipse equation by *ab*) so that the area of the ellipse divided by π appears on the RHS of the defining equation. For a general ellipse

$$Ax^2 + 2Bxy + Cy^2 = D$$

The area is easily computed to be

$$\frac{\text{Area}}{\pi} \equiv \varepsilon = \frac{D}{\sqrt{AC - B^2}} \qquad \text{Eqn. (1)}$$

So the equation is equivalently

$$\gamma x^2 + 2\alpha xy + \beta y^2 = \varepsilon$$

$$\gamma = \frac{A}{\sqrt{AC - B^2}}, \quad \alpha = \frac{B}{\sqrt{AC - B^2}}, \quad \text{and} \quad \beta = \frac{C}{\sqrt{AC - B^2}}$$

When normalized in this manner, the equation coefficients clearly satisfy

$$\beta\gamma - \alpha^2 = 1$$

Example: the defining equation for the upright ellipse may be rewritten in following suggestive way

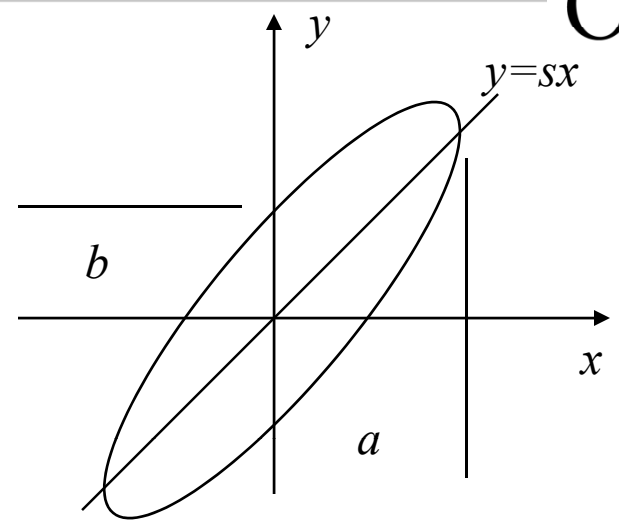
$$\frac{b}{a}x^2 + \frac{a}{b}y^2 = ab = \varepsilon$$

$\beta = a/b$ and $\gamma = b/a$, note $x_{\max} = a = \sqrt{\beta\varepsilon}$, $y_{\max} = b = \sqrt{\gamma\varepsilon}$

General Tilted Ellipse

Needs 3 parameters for a complete description. One way

$$\frac{b}{a}x^2 + \frac{a}{b}(y - sx)^2 = ab = \varepsilon$$



where s is a slope parameter, a is the maximum extent in the x-direction, and the y-intercept occurs at $\pm b$, and again ε is the area of the ellipse divided by π

$$\frac{b}{a}\left(1 + s^2\frac{a^2}{b^2}\right)x^2 - 2s\frac{a}{b}xy + \frac{a}{b}y^2 = ab = \varepsilon$$

Identify

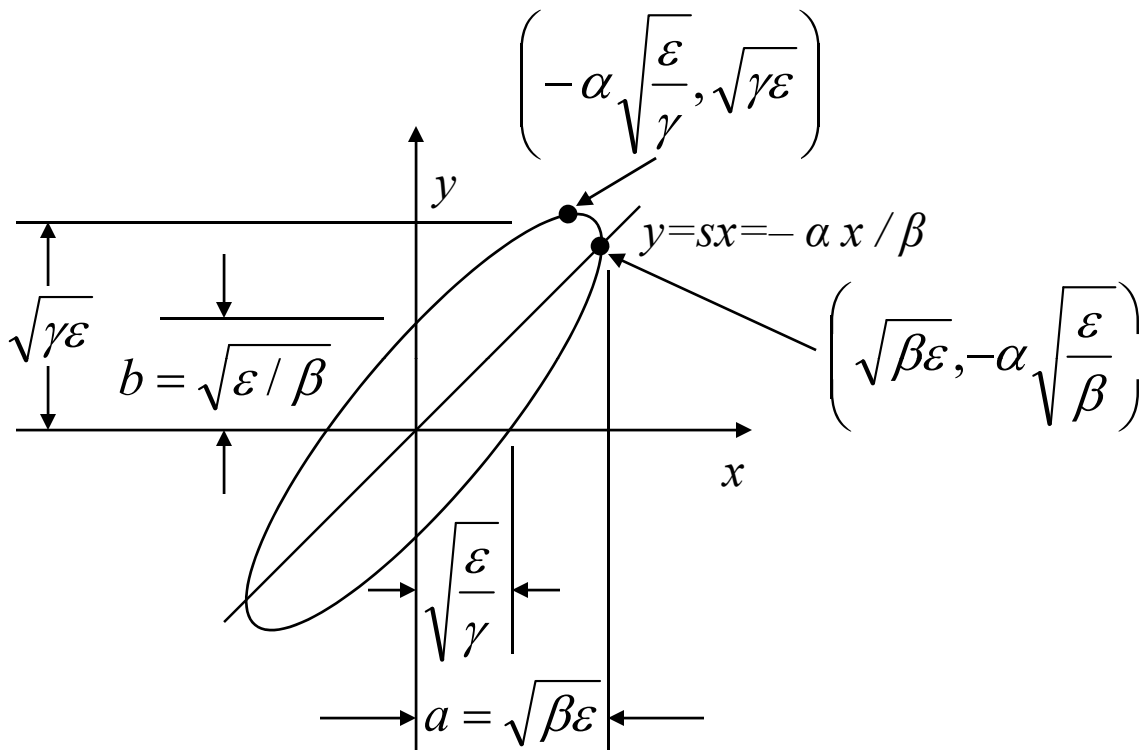
$$\gamma = \frac{b}{a}\left(1 + s^2 \frac{a^2}{b^2}\right), \qquad \alpha = -\frac{a}{b}s, \qquad \beta = \frac{a}{b}$$

Note that $\beta\gamma - \alpha^2 = 1$ automatically, and that the equation for ellipse becomes

$$x^2 + \left(\beta y + \alpha x\right)^2 = \beta\varepsilon$$

by eliminating the (redundant!) parameter γ

Ellipse Dimensions in the β-function Description



As for the upright ellipse $\quad x_{\max} = \sqrt{\beta\varepsilon}, \qquad y_{\max} = \sqrt{\gamma\varepsilon}$

Wille: page 81

Area Theorem for Linear Optics

Under a general linear transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

an ellipse is transformed into another ellipse. Furthermore, if det (M) = 1, the area of the ellipse after the transformation is the same as that before the transformation.

Pf: Let the initial ellipse, normalized as above, be

$$\gamma_0 x^2 + 2\alpha_0 xy + \beta_0 y^2 = \varepsilon_0$$

Because

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \left(M^{-1}\right)_{11} & \left(M^{-1}\right)_{12} \\ \left(M^{-1}\right)_{21} & \left(M^{-1}\right)_{22} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

The transformed ellipse is

$$\gamma x^2 + 2\alpha xy + \beta y^2 = \varepsilon_0$$

$$\gamma = \left(M^{-1}\right)_{11}^2 \gamma_0 + 2\left(M^{-1}\right)_{11}\left(M^{-1}\right)_{21} \alpha_0 + \left(M^{-1}\right)_{21}^2 \beta_0$$

$$\alpha = \left(M^{-1}\right)_{11}\left(M^{-1}\right)_{12} \gamma_0 + \left(\left(M^{-1}\right)_{11}\left(M^{-1}\right)_{22} + \left(M^{-1}\right)_{12}\left(M^{-1}\right)_{21}\right)\alpha_0 + \left(M^{-1}\right)_{21}\left(M^{-1}\right)_{22} \beta_0$$

$$\beta = \left(M^{-1}\right)_{12}^2 \gamma_0 + 2\left(M^{-1}\right)_{12}\left(M^{-1}\right)_{22} \alpha_0 + \left(M^{-1}\right)_{22}^2 \beta_0$$

Because (verify!)

$$\beta\gamma - \alpha^2 = \left(\beta_0\gamma_0 - \alpha_0^2\right)$$
$$\times\left(\left(M^{-1}\right)_{21}^2\left(M^{-1}\right)_{12}^2 + \left(M^{-1}\right)_{11}^2\left(M^{-1}\right)_{22}^2 - 2\left(M^{-1}\right)_{11}\left(M^{-1}\right)_{22}\left(M^{-1}\right)_{12}\left(M^{-1}\right)_{21}\right)$$
$$= \left(\beta_0\gamma_0 - \alpha_0^2\right)\left(\det M^{-1}\right)^2$$

the area of the transformed ellipse (divided by π) is, by Eqn. (1)

$$\frac{\text{Area}}{\pi} = \varepsilon = \frac{\varepsilon_0}{\sqrt{\beta_0\gamma_0 - \alpha_0^2}\left|\det M^{-1}\right|} = \varepsilon_0 \,|\det M|$$

Tilted ellipse from the upright ellipse

In the tilted ellipse the y-coordinate is raised by the slope with respect to the un-tilted ellipse

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\gamma_0 = \frac{b}{a}, \qquad \alpha_0 = 0, \qquad \beta_0 = \frac{a}{b}, \qquad \left(M^{-1}\right)_{21} = -s$$

$$\therefore \quad \gamma = \frac{b}{a} + \frac{a}{b} s^2, \qquad \alpha = -\frac{a}{b} s, \qquad \beta = \frac{a}{b}$$

Because det (M)=1, the tilted ellipse has the same area as the upright ellipse, i.e., $\varepsilon = \varepsilon_0$.

Phase Advance of a Unimodular Matrix

Any two-by-two unimodular (Det (M) = 1) matrix with $|\mathrm{Tr}\, M| < 2$ can be written in the form

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(\mu) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin(\mu)$$

The *phase advance* of the matrix, μ, gives the eigenvalues of the matrix $\lambda = e^{\pm i\mu}$, and $\cos \mu = (\mathrm{Tr}\, M)/2$. Furthermore $\beta\gamma - \alpha^2 = 1$

Pf: The equation for the eigenvalues of M is

$$\lambda^2 - (M_{11} + M_{22})\lambda + 1 = 0$$

Because M is real, both λ and λ^* are solutions of the quadratic. Because

$$\lambda = \frac{\mathrm{Tr}(M)}{2} \pm i\sqrt{1-(\mathrm{Tr}(M)/2)^2}$$

For $|\mathrm{Tr}\, M| < 2$, $\lambda\,\lambda^* = 1$ and so $\lambda_{1,2} = e^{\pm i\mu}$. Consequently $\cos\mu = (\mathrm{Tr}\, M)/2$. Now the following matrix is trace-free.

$$M - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(\mu) = \begin{pmatrix} \frac{M_{11}-M_{22}}{2} & M_{12} \\ M_{21} & \frac{M_{22}-M_{11}}{2} \end{pmatrix}$$

Simply choose

$$\alpha = \frac{M_{11} - M_{22}}{2\sin\mu}, \qquad \beta = \frac{M_{12}}{\sin\mu}, \qquad \gamma = -\frac{M_{21}}{\sin\mu}$$

and the sign of μ to properly match the individual matrix elements with $\beta > 0$. It is easily verified that $\beta\gamma - \alpha^2 = 1$. Now

$$M^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(2\mu) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}\sin(2\mu)$$

and more generally

$$M^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(n\mu) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}\sin(n\mu)$$

Therefore, because sin and cos are both bounded functions, the matrix elements of any power of M remain bounded as long as $|\mathrm{Tr}\,(M)| < 2$.

NB, in some beam dynamics literature it is (incorrectly!) stated that the less stringent $|\mathrm{Tr}\,(M)| \leq 2$ ensures boundedness and/or stability. That equality cannot be allowed can be immediately demonstrated by counterexample. The upper triangular or lower triangular subgroups of the two-by-two unimodular matrices, i.e., matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$$

clearly have unbounded powers if $|x|$ is not equal to 0.

Significance of matrix parameters

Another way to interpret the parameters α, β, and γ, which represent the unimodular matrix M (these parameters are sometimes called the Twiss parameters or Twiss representation for the matrix) is as the "coordinates" of that specific set of ellipses that are mapped onto each other, or are invariant, under the linear action of the matrix. This result is demonstrated in

Thm: For the unimodular linear transformation

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(\mu) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin(\mu)$$

with $|\mathrm{Tr}\,(M)| < 2$, the ellipses

$$\gamma x^2 + 2\alpha xy + \beta y^2 = c$$

are invariant under the linear action of M, where c is any constant. Furthermore, these are the only invariant ellipses. Note that the theorem does not apply to $\pm I$, because $|\mathrm{Tr}\,(\pm I)| = 2$.

Pf: The inverse to M is clearly

$$M^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(\mu) - \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin(\mu)$$

By the ellipse transformation formulas, for example

$$\beta' = \beta^2 \left(\sin^2 \mu\right)\gamma + 2\left(-\beta \sin \mu\right)\left(\cos \mu + \alpha \sin \mu\right)\alpha + \left(\cos \mu + \alpha \sin \mu\right)^2 \beta$$

$$= \beta \sin^2 \mu \left(1 + \alpha^2\right) - 2\beta\alpha^2 \sin^2 \mu + \beta \cos^2 \mu + \beta\alpha^2 \sin^2 \mu$$

$$= \left(\sin^2 \mu + \cos^2 \mu\right)\beta = \beta$$

Similar calculations demonstrate that $\alpha' = \alpha$ and $\gamma' = \gamma$. As det $(M) = 1$, $c' = c$, and therefore the ellipse is invariant. Conversely, suppose that an ellipse is invariant. By the ellipse transformation formula, the specific ellipse

$$\gamma_i x^2 + 2\alpha_i xy + \beta_i y^2 = \varepsilon$$

is invariant under the transformation by M only if

$$\begin{pmatrix} \gamma_i \\ \alpha_i \\ \beta_i \end{pmatrix} = \begin{pmatrix} (\cos\mu - \alpha\sin\mu)^2 & 2(\cos\mu - \alpha\sin\mu)(\gamma\sin\mu) & (\gamma\sin\mu)^2 \\ -(\cos\mu - \alpha\sin\mu)(\beta\sin\mu) & 1 - 2\beta\gamma\sin^2\mu & (\cos\mu + \alpha\sin\mu)(\gamma\sin\mu) \\ (\beta\sin\mu)^2 & -2(\cos\mu + \alpha\sin\mu)(\beta\sin\mu) & (\cos\mu + \alpha\sin\mu)^2 \end{pmatrix} \begin{pmatrix} \gamma_i \\ \alpha_i \\ \beta_i \end{pmatrix}$$

$$\equiv T_M \begin{pmatrix} \gamma_i \\ \alpha_i \\ \beta_i \end{pmatrix} \equiv T_M \vec{v},$$

i.e., if the vector $\vec{v}$ is *ANY* eigenvector of T_M with eigenvalue 1. All possible solutions may be obtained by investigating the eigenvalues and eigenvectors of T_M. Now

$$T_M \vec{v}_\lambda = \lambda \vec{v}_\lambda \quad \text{has a solution when Det}\left(T_M - \lambda I\right) = 0$$

i.e.,

$$\left(\lambda^2 + \left[2 - 4\cos^2\mu\right]\lambda + 1\right)\left(1 - \lambda\right) = 0$$

Therefore, M generates a transformation matrix T_M with at least one eigenvalue equal to 1. For there to be more than one solution with $\lambda = 1$,

$$1 + \left[2 - 4\cos^2\mu\right] + 1 = 0, \quad \cos^2\mu = 1, \quad \text{or } M = \pm I$$

and we note that *all* ellipses are invariant when $M = \pm I$. But, these two cases are excluded by hypothesis. Therefore, M generates a transformation matrix T_M which always possesses a single nondegenerate eigenvalue 1; the set of eigenvectors corresponding to the eigenvalue 1, all proportional to each other, are the only vectors whose components (γ_i, α_i, β_i) yield equations for the invariant ellipses. For concreteness, compute that eigenvector with eigenvalue 1 normalized so $\beta_i\gamma_i - \alpha_i^2 = 1$

$$\vec{v}_{1,i} = \begin{pmatrix} \gamma_i \\ \alpha_i \\ \beta_i \end{pmatrix} = \beta \begin{pmatrix} -M_{21}/M_{12} \\ (M_{11} - M_{22})/2M_{12} \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma \\ \alpha \\ \beta \end{pmatrix}$$

All other eigenvectors with eigenvalue 1 have $\vec{v}_1 = \varepsilon \vec{v}_{1,i} / c$, for some value c.

Because Det (M) =1, the eigenvector $\vec{v}_{1,i}$ clearly yields the invariant ellipse

$$\gamma x^2 + 2\alpha xy + \beta y^2 = \varepsilon.$$

Likewise, the proportional eigenvector $\vec{v}_1$ generates the similar ellipse

$$\frac{\varepsilon}{c}\left(\gamma x^2 + 2\alpha xy + \beta y^2\right) = \varepsilon$$

Because we have enumerated all possible eigenvectors with eigenvalue 1, all ellipses invariant under the action of M, are of the form

$$\gamma x^2 + 2\alpha xy + \beta y^2 = c$$

To summarize, this theorem gives a way to tie the mathematical representation of a unimodular matrix in terms of its α, β, and γ, and its phase advance, to the equations of the ellipses invariant under the matrix transformation. The equations of the invariant ellipses when properly normalized have precisely the same α, β, and γ as in the Twiss representation of the matrix, but varying c.

Finally note that throughout this calculation c acts merely as a scale parameter for the ellipse. All ellipses similar to the starting ellipse, i.e., ellipses whose equations have the same α, β, and γ, but with different c, are also invariant under the action of M. Later, it will be shown that more generally

$$\varepsilon = \gamma x^2 + 2\alpha x x' + \beta x'^2 = \left(x^2 + \left(\beta x' + \alpha x\right)^2\right) / \beta$$

is an invariant of the equations of transverse motion.

Applications to transverse beam optics

When the motion of particles in transverse phase space is considered, linear optics provides a good first approximation of the transverse particle motion. Beams of particles are represented by ellipses in phase space (i.e. in the (x, x') space). To the extent that the transverse forces are linear in the deviation of the particles from some pre-defined central orbit, the motion may analyzed by applying ellipse transformation techniques.

Transverse Optics Conventions: positions are measured in terms of length and angles are measured by radian measure. The area in phase space divided by π, ε, measured in m-rad, is called the emittance. In such applications, α has no units, β has units m/radian. Codes that calculate β, by widely accepted convention, drop the per radian when reporting results, it is implicit that the units for x' are radians.

Linear Transport Matrix

Within a linear optics description of transverse particle motion, the particle transverse coordinates at a location s along the beam line are described by a vector

$$\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix}$$

If the differential equation giving the evolution of x is linear, one may define a linear transport matrix $M_{s',s}$ relating the coordinates at s' to those at s by

$$\begin{pmatrix} x(s') \\ \frac{dx}{ds}(s') \end{pmatrix} = M_{s',s} \begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix}$$

From the definitions, the concatenation rule $M_{s'',s} = M_{s'',s'} M_{s',s}$ must apply for all s' such that $s < s' < s''$ where the multiplication is the usual matrix multiplication.

Pf: The equations of motion, linear in x and dx/ds, generate a motion with

$$M_{s'',s}\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} x(s'') \\ \frac{dx}{ds}(s'') \end{pmatrix} = M_{s'',s'}\begin{pmatrix} x(s') \\ \frac{dx}{ds}(s') \end{pmatrix} = M_{s'',s'} M_{s',s}\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix}$$

for all initial conditions ($x(s)$, $dx/ds(s)$), thus $M_{s'',s} = M_{s'',s'} M_{s',s}$.

Clearly $M_{s,s} = I$. As is shown next, the matrix $M_{s',s}$ is in general a member of the unimodular subgroup of the general linear group.

Ellipse Transformations Generated by Hill's Equation

The equation governing the linear transverse dynamics in a particle accelerator, without acceleration, is *Hill's equation**

$$\frac{d^2x}{ds^2} + K(s)x = 0 \qquad \text{Eqn. (2)}$$

The transformation matrix taking a solution through an infinitesimal distance *ds* is

$$\begin{pmatrix} x(s+ds) \\ \frac{dx}{ds}(s+ds) \end{pmatrix} = \begin{pmatrix} 1 & \frac{ds}{\text{rad}} \\ -K(s)ds\ \text{rad} & 1 \end{pmatrix} \begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} \equiv M_{s+ds,s} \begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix}$$

* Strictly speaking, Hill studied Eqn. (2) with periodic *K*. It was first applied to circular accelerators which had a periodicity given by the circumference of the machine. It is a now standard in the field of beam optics, to still refer to Eqn. 2 as Hill's equation, even in cases, as in linear accelerators, where there is no periodicity.

Suppose we are given the phase space ellipse

$$\gamma(s)x^2 + 2\alpha(s)xx' + \beta(s)x'^2 = \varepsilon$$

at location s, and we wish to calculate the ellipse parameters, after the motion generated by Hill's equation, at the location $s + ds$

$$\gamma(s+ds)x^2 + 2\alpha(s+ds)xx' + \beta(s+ds)x'^2 = \varepsilon'$$

Because, to order linear in ds, Det $M_{s+ds,s} = 1$, at all locations s, $\varepsilon' = \varepsilon$, and thus the phase space area of the ellipse after an infinitesimal displacement must equal the phase space area before the displacement. Because the transformation through a finite interval in s can be written as a series of infinitesimal displacement transformations, all of which preserve the phase space area of the transformed ellipse, we come to two important conclusions:

1. The phase space area is preserved after a finite integration of Hill's equation to obtain $M_{s',s}$, the transport matrix which can be used to take an ellipse at s to an ellipse at s'. This conclusion holds generally for all s' and s.

2. Therefore Det $M_{s',s} = 1$ for all s' and s, independent of the details of the functional form $K(s)$. (If desired, these two conclusions may be verified more analytically by showing that

$$\frac{d}{ds}\left(\beta\gamma - \alpha^2\right) = 0 \quad \rightarrow \quad \beta(s)\gamma(s) - \alpha^2(s) = 1, \quad \forall s$$

may be derived directly from Hill's equation.)

Evolution equations for the α, β functions

The ellipse transformation formulas give, to order linear in *ds*

$$\beta(s+ds) = -2\alpha\frac{ds}{\text{rad}} + \beta(s)$$

$$\alpha(s+ds) = -\gamma(s)\frac{ds}{\text{rad}} + \alpha(s) + \beta(s)Kds\ \text{rad}$$

So

$$\frac{d\beta}{ds}(s) = -\frac{2\alpha(s)}{\text{rad}}$$

$$\frac{d\alpha}{ds}(s) = \beta(s)K\ \text{rad} - \frac{\gamma(s)}{\text{rad}}$$

Note that these two formulas are independent of the scale of the starting ellipse ε, and in theory may be integrated directly for $\beta(s)$ and $\alpha(s)$ given the focusing function $K(s)$. A somewhat easier approach to obtain $\beta(s)$ is to recall that the maximum extent of an ellipse, $x_{\max}$, is $(\varepsilon\beta)^{1/2}(s)$, and to solve the differential equation describing its evolution. The above equations may be combined to give the following non-linear equation for $x_{\max}(s) = w(s) = (\varepsilon\beta)^{1/2}(s)$

$$\frac{d^2 w}{ds^2} + K(s)\, w = \frac{(\varepsilon / \mathrm{rad})^2}{w^3}.$$

Such a differential equation describing the evolution of the maximum extent of an ellipse being transformed is known as an *envelope equation*.

It should be noted, for consistency, that the same $\beta(s) = w^2(s)/\varepsilon$ is obtained if one starts integrating the ellipse evolution equation from a different, but similar, starting ellipse. That this is so is an exercise.

The envelope equation may be solved with the correct boundary conditions, to obtain the β-function. α may then be obtained from the derivative of β, and γ by the usual normalization formula. Types of boundary conditions: Class I—periodic boundary conditions suitable for circular machines or periodic focusing lattices, Class II—initial condition boundary conditions suitable for linacs or recirculating machines.

Solution to Hill's Equation in Amplitude-Phase form

To get a more general expression for the phase advance, consider in more detail the single particle solutions to Hill's equation

$$\frac{d^2x}{ds^2} + K(s)x = 0$$

From the theory of linear ODEs, the general solution of Hill's equation can be written as the sum of the two linearly independent pseudo-harmonic functions

$$x(s) = Ax_+(s) + Bx_-(s)$$

where

$$x_\pm(s) = w(s)e^{\pm i\mu(s)}$$

are two particular solutions to Hill's equation, provided that

$$\frac{d^2 w}{ds^2} + K(s)w = \frac{c^2}{w^3} \qquad \text{and} \qquad \frac{d\mu}{ds}(s) = \frac{c}{w^2(s)}, \quad \text{Eqns. (3)}$$

and where A, B, and c are constants (in s)

That specific solution with boundary conditions $x(s_1) = x_1$ and $dx/ds\ (s_1) = x'_1$ has

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} w(s_1)e^{i\mu(s_1)} & w(s_1)e^{-i\mu(s_1)} \\ \left[w'(s_1) + \frac{ic}{w(s_1)}\right]e^{i\mu(s_1)} & \left[w'(s_1) - \frac{ic}{w(s_1)}\right]e^{-i\mu(s_1)} \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}$$

Therefore, the unimodular transfer matrix taking the solution at $s = s_1$ to its coordinates at $s = s_2$ is

$$\begin{pmatrix} x_2 \\ x'_2 \end{pmatrix} = \begin{pmatrix} \frac{w(s_2)}{w(s_1)}\cos\Delta\mu_{s_2,s_1} - \frac{w(s_2)w'(s_1)}{c}\sin\Delta\mu_{s_2,s_1} & \frac{w(s_2)w(s_1)}{c}\sin\Delta\mu_{s_2,s_1} \\ -\frac{c}{w(s_2)w(s_1)}\left[1+\frac{w(s_2)w'(s_2)w(s_1)w'(s_1)}{c^2}\right]\sin\Delta\mu_{s_2,s_1} & \frac{w(s_1)}{w(s_2)}\cos\Delta\mu_{s_2,s_1} + \frac{w'(s_2)w(s_1)}{c}\sin\Delta\mu_{s_2,s_1} \\ -\left[\frac{w'(s_1)}{w(s_2)} - \frac{w'(s_2)}{w(s_1)}\right]\cos\Delta\mu_{s_2,s_1} & \end{pmatrix} \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}$$

where

$$\Delta\mu_{s_2,s_1} = \mu(s_2) - \mu(s_1) = \int_{s_1}^{s_2} \frac{c}{w^2(s)} ds$$

Case I: $K(s)$ periodic in s

Such boundary conditions, which may be used to describe circular or ring-like accelerators, or periodic focusing lattices, have $K(s + L) = K(s)$. L is either the machine circumference or period length of the focusing lattice.

It is natural to assume that there exists a unique periodic solution $w(s)$ to Eqn. (3a) when $K(s)$ is periodic. Here, we will assume this to be the case. Later, it will be shown how to construct the function explicitly. Clearly for w periodic

$$\phi(s) = \mu(s) - \mu_L s \qquad \text{with} \qquad \mu_L = \int_s^{s+L} \frac{c}{w^2(s)} ds$$

is also periodic by Eqn. (3b), and μ_L is independent of s.

The transfer matrix for a single period reduces to

$$\begin{pmatrix} \cos\mu_L - \dfrac{w(s)w'(s)}{c}\sin\mu_L & \dfrac{w^2(s)}{c}\sin\mu_L \\ -\dfrac{c}{w^2(s)}\left[1+\dfrac{w(s)w'(s)w(s)w'(s)}{c^2}\right]\sin\mu_L & \cos\mu_L + \dfrac{w'(s)w(s)}{c}\sin\mu_L \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(\mu_L) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}\sin(\mu_L)$$

where the (now periodic!) matrix functions are

$$\alpha(s) = -\frac{w(s)w'(s)}{c}, \quad \beta(s) = \frac{w^2(s)}{c}, \quad \gamma(s) = \frac{1+\alpha^2(s)}{\beta(s)}$$

By Thm. (2), these are the ellipse parameters of the periodically repeating, i.e., *matched* ellipses.

General formula for phase advance

In terms of the β-function, the phase advance for the period is

$$\mu_L = \int_0^L \frac{ds}{\beta(s)}$$

and more generally the phase advance between any two longitudinal locations s and s' is

$$\Delta\mu_{s',s} = \int_s^{s'} \frac{ds}{\beta(s)}$$

Transfer Matrix in terms of α and β

Also, the unimodular transfer matrix taking the solution from s to s' is

$$M_{s',s} = \begin{pmatrix} \sqrt{\frac{\beta(s')}{\beta(s)}}\left(\cos\Delta\mu_{s',s} + \alpha(s)\sin\Delta\mu_{s',s}\right) & \sqrt{\beta(s')\beta(s)}\sin\Delta\mu_{s',s} \\ -\frac{1}{\sqrt{\beta(s')\beta(s)}}\begin{bmatrix}(1+\alpha(s')\alpha(s))\sin\Delta\mu_{s',s} \\ +(\alpha(s')-\alpha(s))\cos\Delta\mu_{s',s}\end{bmatrix} & \sqrt{\frac{\beta(s)}{\beta(s')}}\left(\cos\Delta\mu_{s',s} - \alpha(s')\sin\Delta\mu_{s',s}\right) \end{pmatrix}$$

Note that this final transfer matrix and the final expression for the phase advance do not depend on the constant c. This conclusion might have been anticipated because different particular solutions to Hill's equation exist for all values of c, but from the theory of linear ordinary differential equations, the final motion is unique once x and dx/ds are specified somewhere.

Method to compute the β-function

Our previous work has indicated a method to compute the β-function (and thus w) directly, i.e., without solving the differential equation Eqn. (3). At a given location s, determine the one-period transfer map $M_{s+L,s}(s)$. From this find μ_L (which is independent of the location chosen!) from $\cos \mu_L = (M_{11}+M_{22}) / 2$, and by choosing the sign of μ_L so that $\beta(s) = M_{12}(s) / \sin \mu_L$ is positive. Likewise, $\alpha(s) = (M_{11}-M_{22}) / 2 \sin \mu_L$. Repeat this exercise at every location the β-function is desired.

By construction, the beta-function and the alpha-function, and hence w, are periodic because the single-period transfer map is periodic. It is straightforward to show $w=(c\beta(s))^{1/2}$ satisfies the envelope equation.

Courant-Snyder Invariant

Consider now a single particular solution of the equations of motion generated by Hill's equation. We've seen that once a particle is on an invariant ellipse for a period, it must stay on that ellipse throughout its motion. Because the phase space area of the single period invariant ellipse is preserved by the motion, the quantity that gives the phase space area of the invariant ellipse in terms of the single particle orbit must also be an invariant. This phase space area/π,

$$\varepsilon = \gamma x^2 + 2\alpha x x' + \beta x'^2 = \left(x^2 + \left(\beta x' + \alpha x\right)^2\right) / \beta$$

is called the Courant-Snyder invariant. It may be verified to be a constant by showing its derivative with respect to s is zero by Hill's equation, or by explicit substitution of the transfer matrix solution which begins at some initial value $s = 0$.

Pseudoharmonic Solution

$$\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta_0}}\left(\cos\Delta\mu_{s,0} + \alpha_0 \sin\Delta\mu_{s,0}\right) & \sqrt{\beta(s)\beta_0}\sin\Delta\mu_{s,0} \\ -\frac{1}{\sqrt{\beta(s)\beta_0}}\begin{bmatrix}(1+\alpha(s)\alpha_0)\sin\Delta\mu_{s,0} \\ +(\alpha(s)-\alpha_0)\cos\Delta\mu_{s,0}\end{bmatrix} & \sqrt{\frac{\beta_0}{\beta(s)}}\left(\cos\Delta\mu_{s,0} - \alpha(s)\sin\Delta\mu_{s,0}\right) \end{pmatrix} \begin{pmatrix} x_0 \\ \left.\frac{dx}{ds}\right|_0 \end{pmatrix}$$

gives

$$\left(x^2(s) + \left(\beta(s)x'(s) + \alpha(s)x(s)\right)^2\right)/\beta(s) = \left(x_0^2 + \left(\beta_0 x'_0 + \alpha_0 x_0\right)^2\right)/\beta_0 \equiv \varepsilon$$

Using the $x(s)$ equation above and the definition of ε, the solution may be written in the standard "pseudoharmonic" form

$$x(s) = \sqrt{\varepsilon\beta(s)}\cos\left(\Delta\mu_{s,0} - \delta\right) \quad \text{where } \delta = \tan^{-1}\left(\frac{\beta_0 x'_0 + \alpha_0 x_0}{x_0}\right)$$

The the origin of the terminology "phase advance" is now obvious.

Case II: $K(s)$ not periodic

In a linac or a recirculating linac there is no closed orbit or natural machine periodicity. Designing the transverse optics consists of arranging a focusing lattice that assures the beam particles coming into the front end of the accelerator are accelerated (and sometimes decelerated!) with as small beam loss as is possible. Therefore, it is imperative to know the initial beam phase space injected into the accelerator, in addition to the transfer matrices of all the elements making up the focusing lattice of the machine. An initial ellipse, or a set of initial conditions that somehow bound the phase space of the injected beam, are tracked through the acceleration system element by element to determine the transmission of the beam through the accelerator. The designs are usually made up of well-understood "modules" that yield known and understood transverse beam optical properties.

Definition of β function

Now the pseudoharmonic solution applies even when $K(s)$ is not periodic. Suppose there is an ellipse, the design injected ellipse, which tightly includes the phase space of the beam at injection to the accelerator. Let the ellipse parameters for this ellipse be α_0, β_0, and γ_0. A function $\beta(s)$ is simply defined by the ellipse transformation rule

$$\begin{aligned}\beta(s) &= (M_{12}(s))^2\gamma_0 - 2M_{12}(s)M_{11}(s)\alpha_0 + (M_{11}(s))^2\beta_0 \\ &= \left[(M_{12}(s))^2 + (\beta_0 M_{11}(s) - \alpha_0 M_{12}(s))^2\right]/\beta_0\end{aligned}$$

where

$$M_{s,0} \equiv \begin{pmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{pmatrix}$$

One might think to evaluate the phase advance by integrating the beta-function. Generally, it is far easier to evaluate the phase advance using the general formula,

$$\tan \Delta\mu_{s',s} = \frac{\left(M_{s',s}\right)_{12}}{\beta(s)\left(M_{s',s}\right)_{11} - \alpha(s)\left(M_{s',s}\right)_{12}}$$

where $\beta(s)$ and $\alpha(s)$ are the ellipse functions at the entrance of the region described by transport matrix $M_{s',s}$. Applied to the situation at hand yields

$$\tan \Delta\mu_{s,0} = \frac{M_{12}(s)}{\beta_0 M_{11}(s) - \alpha_0 M_{12}(s)}$$

Beam Matching

Fundamentally, in circular accelerators beam matching is applied in order to guarantee that the beam envelope of the real accelerator beam does not depend on time. This requirement is one part of the definition of having a stable beam. With periodic boundary conditions, this means making beam density contours in phase space align with the invariant ellipses (in particular at the injection location!) given by the ellipse functions. Once the particles are on the invariant ellipses they stay there (in the linear approximation!), and the density is preserved because the single particle motion is around the invariant ellipses. In linacs and recirculating linacs, usually different purposes are to be achieved. If there are regions with periodic focusing lattices within the linacs, matching as above ensures that the beam

envelope does not grow going down the lattice. Sometimes it is advantageous to have specific values of the ellipse functions at specific longitudinal locations. Other times, re/matching is done to preserve the beam envelopes of a good beam solution as changes in the lattice are made to achieve other purposes, e.g. changing the dispersion function or changing the chromaticity of regions where there are bends (see the next chapter for definitions). At a minimum, there is usually a matching done in the first parts of the injector, to take the phase space that is generated by the particle source, and change this phase space in a way towards agreement with the nominal transverse focusing design of the rest of the accelerator. The ellipse transformation formulas, solved by computer, are essential for performing this process.

Dispersion Calculation

Begin with the inhomogeneous Hill's equation for the dispersion.

$$\frac{d^2 D}{ds^2} + K(s) D = \frac{1}{\rho(s)}$$

Write the general solution to the inhomogeneous equation for the dispersion as before.

$$D(s) = D_p(s) + A x_1(s) + B x_2(s)$$

Here D_p can be any particular solution. Suppose that the dispersion and it's derivative are known at the location s_1, and we wish to determine their values at s_2. x_1 and x_2, because they are solutions to the homogeneous equations, must be transported by the transfer matrix solution $M_{s2,s1}$ already found.

To build up the general solution, choose that particular solution of the inhomogeneous equation with boundary conditions

$$D_{p,0}\left(s_1\right) = D'_{p,0}\left(s_1\right) = 0$$

Evaluate A and B by the requirement that the dispersion and it's derivative have the proper value at s_1 (x_1 and x_2 need to be linearly independent!)

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} x_1\left(s_1\right) & x_2\left(s_1\right) \\ x'_1\left(s_1\right) & x'_2\left(s_1\right) \end{pmatrix}^{-1} \begin{pmatrix} D\left(s_1\right) \\ D'\left(s_1\right) \end{pmatrix}$$

$$D\left(s_2\right) = D_{p,0}\left(s_2 - s_1\right) + \left(M_{s_2,s_1}\right)_{11} D\left(s_1\right) + \left(M_{s_2,s_1}\right)_{12} D'\left(s_1\right)$$

$$D'\left(s_2\right) = D'_{p,0}\left(s_2 - s_1\right) + \left(M_{s_2,s_1}\right)_{21} D\left(s_1\right) + \left(M_{s_2,s_1}\right)_{22} D'\left(s_1\right)$$

3 by 3 Matrices for Dispersion Tracking

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} \left(M_{s_2,s_1}\right)_{11} & \left(M_{s_2,s_1}\right)_{12} & D_{p,0}(s_2 - s_1) \\ \left(M_{s_2,s_1}\right)_{21} & \left(M_{s_2,s_1}\right)_{22} & D'_{p,0}(s_2 - s_1) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}$$

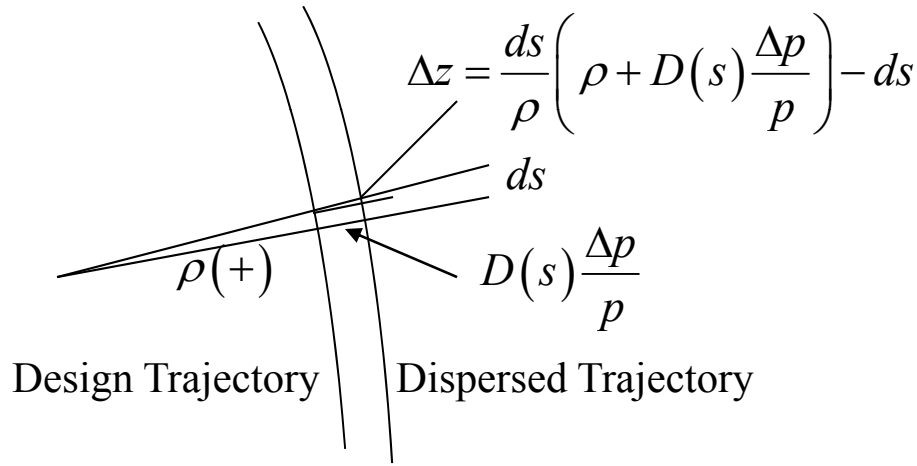
Particular solutions to inhomogeneous equation for constant K and constant ρ and vanishing dispersion and derivative at $s = 0$

	$K < 0$	$K = 0$	$K > 0$
$D_{p,0}(s)$	$\frac{1}{\lvert K\rvert\rho}\left(\cosh\left(\sqrt{\lvert K\rvert}s\right)-1\right)$	$\frac{s^2}{2\rho}$	$\frac{1}{K\rho}\left(1-\cos\left(\sqrt{K}s\right)\right)$
$D'_{p,0}(s)$	$\frac{1}{\sqrt{\lvert K\rvert}\rho}\sinh\left(\sqrt{\lvert K\rvert}s\right)$	$\frac{s}{\rho}$	$\frac{1}{\sqrt{K}\rho}\sin\left(\sqrt{K}s\right)$

M_{56}

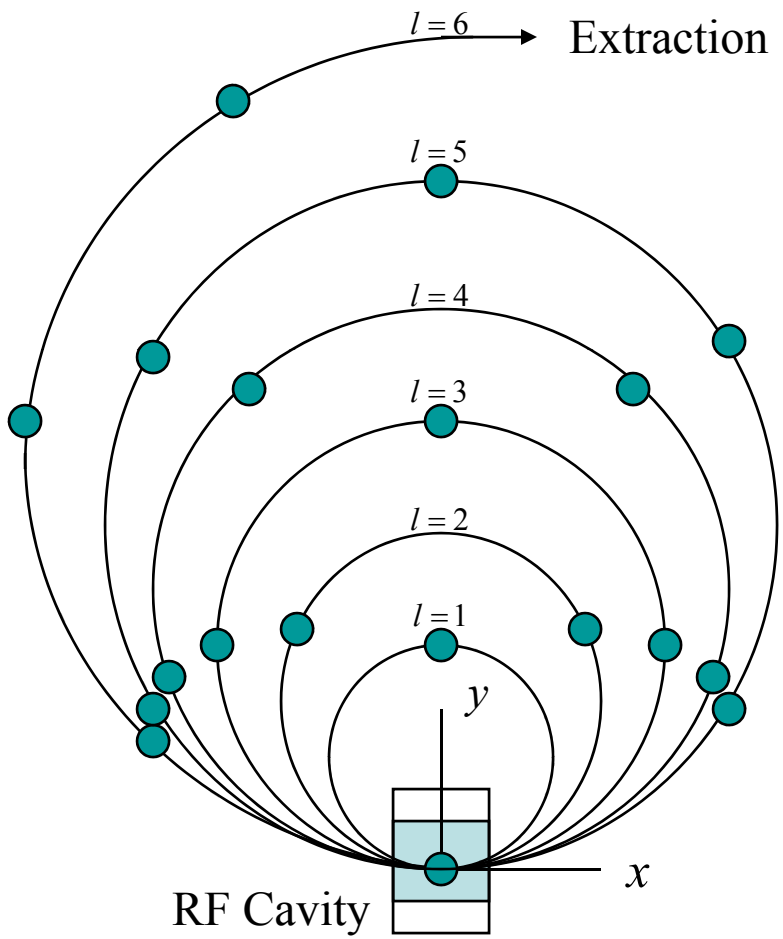
In addition to the transverse effects of the dispersion, there are important effects of the dispersion along the direction of motion. The primary effect is to change the time-of-arrival of the off-momentum particle compared to the on-momentum particle which traverses the design trajectory.

$$d(\Delta z) = D(s)\frac{\Delta p}{p}\frac{ds}{\rho(s)}$$



$$M_{56} = \int_{s_1}^{s_2} \left\{ \frac{D_x(s)}{\rho_x(s)} + \frac{D_y(s)}{\rho_y(s)} \right\} ds$$

Classical Microtron: Veksler (1945)



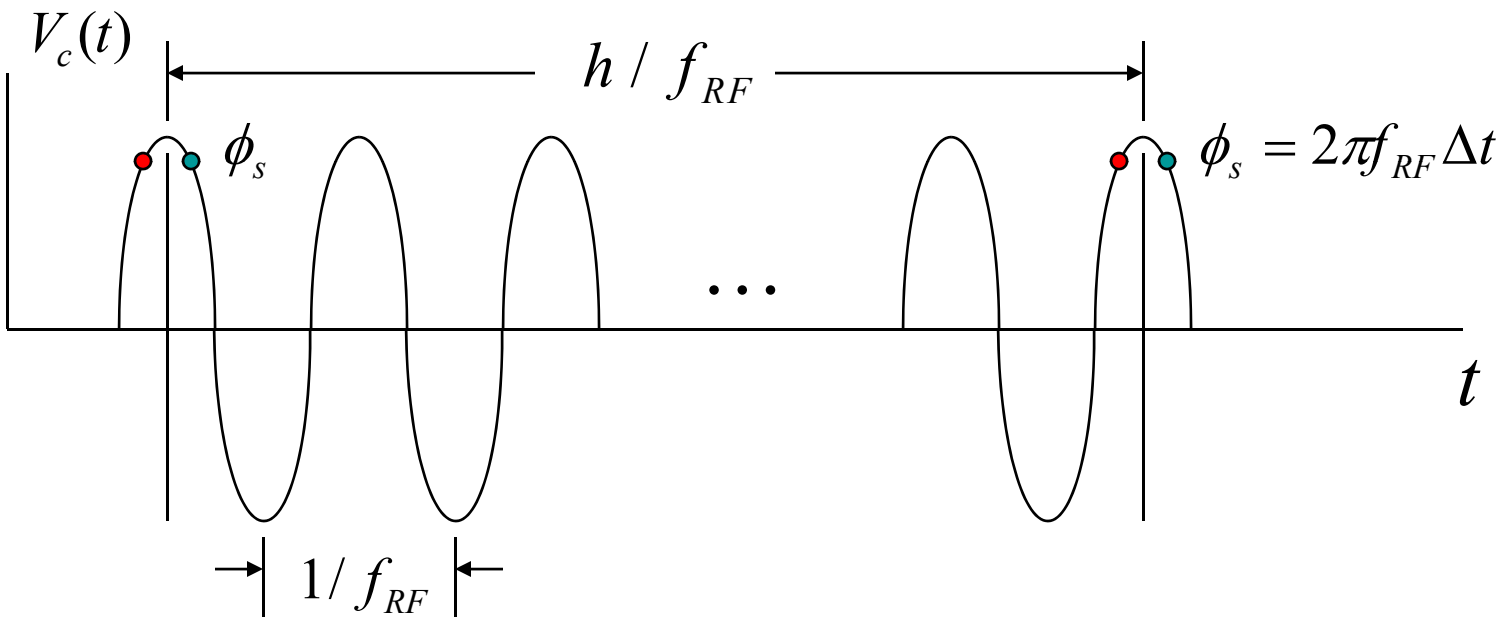
$\otimes$ Magnetic Field

$\mu = 2$

$\nu = 1$

Synchrotron Phase Stability

Edwin McMillan discovered phase stability independently of Veksler and used the idea to design first large electron synchrotron.



$$h = L f_{RF} / \beta c$$

Harmonic number: # of RF oscillations in a revolution

Transition Energy

Beam energy where speed increment effect balances path length change effect on accelerator revolution frequency. Revolution frequency independent of beam energy to linear order. We will calculate in a few weeks

- Below Transistion Energy: Particles arriving EARLY get less acceleration and speed increment, and arrive later, with repect to the center of the bunch, on the next pass. Applies to heavy particle synchrotrons during first part of acceleration when the beam is non-relativistic and accelerations still produce velocity changes.

- Above Transistion Energy: Particles arriving EARLY get more energy, have a longer path, and arrive later on the next pass. Applies for electron synchrotrons and heavy particle synchrotrons when approach relativistic velocities. As seen before, Microtrons operate here.

Phase Stability Condition

"Synchronous" electron has

$$\text{Phase} = \phi_s \qquad\qquad E_l = E_o + leV_c \cos\phi_s$$

Difference equation for differences after passing through cavity pass $l+1$:

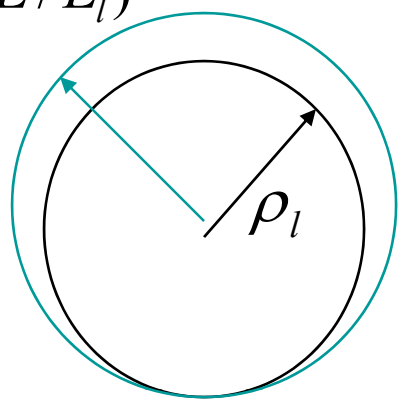
$$\begin{pmatrix} \Delta\phi_{l+1} \\ \Delta E_{l+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -eV_c \sin\phi_s & 1 \end{pmatrix} \begin{pmatrix} 1 & \dfrac{2\pi M_{56}}{\lambda E_l} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta\phi_l \\ \Delta E_l \end{pmatrix}$$

Because for an electron passing the cavity

$$\Delta E_{after} = \Delta E_{before} + eV_c\left(\cos(\phi_s + \Delta\phi) - \cos\phi_s\right)$$

Phase Stability Condition

$\rho_l(1+\Delta E/E_l)$



$$K_i = 1/\rho_i^2$$

$$D_{x,p,0} = \rho_i\left(1-\cos\left(s/\rho_i\right)\right) \qquad 0 \le s \le 2\pi\rho_i$$

$$\therefore\ M_{56} = \int \frac{D}{\rho} ds = \int_0^{2\pi\rho_l} \left(1-\cos s/\rho_l\right) ds$$

$$= 2\pi\rho_l$$

$$\begin{pmatrix} \Delta\phi_{l+1} \\ \Delta E_{l+1} \end{pmatrix} \approx \begin{pmatrix} 1 & \dfrac{4\pi^2\rho_l}{\lambda E_l} \\ -eV_c\sin\phi_s & 1-\dfrac{4\pi^2\rho_l eV_c}{\lambda E_l}\sin\phi_s \end{pmatrix} \begin{pmatrix} \Delta\phi_l \\ \Delta E_l \end{pmatrix}$$

Phase Stability Condition

Have Phase Stability if

$$-1 < \left(\frac{\mathrm{Tr}\, M}{2} \right) < 1 \rightarrow -1 < 1 - \frac{2\pi^2 \rho_l e V_c}{\lambda E_l} \sin\phi_s < 1$$

$$\frac{2\pi^2 \rho_l e V_c}{\lambda E_l} \sin\phi_s = \frac{\pi f_{RF} e V_c}{f_c m c^2} \cos\phi_s \tan\phi_s = \frac{\pi f_{RF} \Delta\gamma}{f_c} \tan\phi_s$$

i.e.,

$$\boxed{0 < \nu\pi \tan\phi_s < 2}$$

Phase Stability Condition

Have Phase Stability if

$$\left(\frac{\mathrm{Tr}\, M}{2}\right)^2 < 1$$

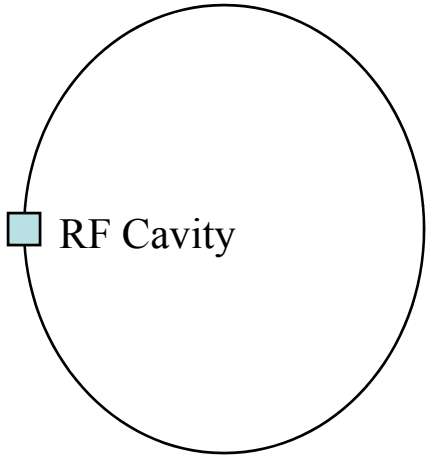
i.e.,

$$\boxed{0 < \nu\pi \tan\phi_s < 2}$$

Synchrotrons

Two basic generalizations needed

- Acceleration of non-relativistic particles
- Difference equation describing per turn dynamics becomes a differential equation with solution involving a new frequency, the synchrotron frequency



Acceleration of non-relativistic particles

For microtron, racetrack microtron and other polytrons, electron speed is at the speed of light. For non-relativistic particles the recirculation time also depends on the longitudinal velocity $v_z = \beta_z c$.

$$t_{recirc} = L / \beta_z c$$

$$\Delta t = \frac{\partial L}{\partial p}\frac{\Delta p}{\beta_z c} + \frac{L}{c}\frac{\partial}{\partial p}\left[\frac{1}{\beta_z}\right]\Delta p$$

$$\frac{\Delta t}{t_{recirc}} = \frac{M_{56}}{L}\frac{\Delta p}{p} - \frac{\Delta \beta_z}{\beta_z} = \frac{M_{56}}{L}\frac{\Delta p}{p} - \frac{1}{\gamma^2}\frac{\Delta p}{p}$$

Momentum Compaction $\alpha = \left(\Delta L / L\right) / \left(\Delta p / p\right) = M_{56} / L$

$$\frac{\Delta t}{t_{recirc}} = -\eta_c \frac{\Delta p}{p} \rightarrow \eta_c = \frac{1}{\gamma^2} - \frac{M_{56}}{L} = \frac{1}{\gamma^2} - \alpha$$

$$2p\Delta pc^2 = 2E\Delta E \rightarrow \frac{\Delta p}{p} = \frac{1}{\beta_z^2}\frac{\Delta E}{E} \rightarrow \frac{\Delta t}{t_{recirc}} = -\frac{\eta_c}{\beta_z^2}\frac{\Delta E}{E}$$

Transition Energy: Energy at which the change in the once around time becomes independent of momentum (energy)

$$\eta_c = 0 \rightarrow \frac{1}{\gamma_t^2} = \frac{M_{56}}{L} = \alpha$$

No Phase Focusing at this energy!

Equation for Synchrotron Oscillations

$$\begin{pmatrix} \Delta\phi_{l+1} \\ \Delta E_{l+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -eV_c \sin\phi_s & 1 \end{pmatrix} \begin{pmatrix} 1 & -\dfrac{2\pi L\eta_c}{\lambda\beta_z^2 E_l} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta\phi_l \\ \Delta E_l \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\dfrac{2\pi L\eta_c}{\lambda\beta_z^2 E_l} \\ -eV_c \sin\phi_s & 1 + \dfrac{2\pi L\eta_c}{\lambda\beta_z^2 E_l} eV_c \sin\phi_s \end{pmatrix} \begin{pmatrix} \Delta\phi_l \\ \Delta E_l \end{pmatrix}$$

Assume momentum slowly changing (adiabatic acceleration)
Phase advance per turn is

$$\cos\Delta\mu = 1 + \frac{\pi L\eta_c}{\lambda\beta_z^2 E_l} eV_c \sin\phi_s \rightarrow \Delta\mu^2 \approx -\frac{2\pi L\eta_c}{\lambda\beta_z^2 E_l} eV_c \sin\phi_s$$

So change in phase per unit time is

$$\frac{\Delta\mu}{T_0} \approx \frac{1}{T_0}\sqrt{-\frac{2\pi L\eta_c}{\lambda\beta_z pc} eV_c \sin\phi_s}$$

yielding synchrotron oscillations with frequency

$$\omega_s = \omega_{rev}\sqrt{-\frac{h\eta_c}{2\pi}\frac{eV_c}{pc}\sin\phi_s}$$

where the *harmonic number* $h = L / \beta_z \lambda$, gives the integer number of *RF* oscillations in one turn

Phase Stable Acceleration

At energies below transition, $\eta_c > 0$. To achieve acceleration with phase stability need $\phi_s < 0$

$$\therefore \omega_s = \omega_{rev}\sqrt{\frac{h\eta_c}{2\pi}\frac{eV_c}{pc}\sin\left(-\phi_s\right)}$$

At energies above transition, $\eta_c < 0$, which corresponds to the case we're used to from electrons. To achieve acceleration with phase stability need $\phi_s > 0$

$$\therefore \omega_s = \omega_{rev}\sqrt{\frac{h\left(-\eta_c\right)}{2\pi}\frac{eV_c}{pc}\sin\phi_s}$$

Large Amplitude Effects

Can no longer linearize the energy error equation.

$$\Delta\phi_{l+1} = \Delta\phi_l - \frac{2\pi L\eta_c}{\lambda\beta_z^2 E_l}\Delta E_l$$

$$\Delta E_{l+1} = \Delta E_l + eV_c\left(\cos\left(\phi_s + \Delta\phi_l\right) - \cos\phi_s\right)$$

$$\frac{d\Delta\phi}{dt} \approx \frac{\Delta\phi_{l+1} - \Delta\phi_l}{T_0} = -\frac{2\pi\eta_c}{\lambda p}\Delta E$$

$$\frac{d\Delta E}{dt} \approx \frac{\Delta E_{l+1} - \Delta E_l}{T_0} = \frac{eV_c\left(\cos\left(\phi_s + \Delta\phi_l\right) - \cos\phi_s\right)}{T_0}$$

$$\frac{d^2\Delta\phi}{dt^2} = -\frac{2\pi\eta_c}{\lambda p T_0} eV_c\left(\cos\left(\phi_s + \Delta\phi\right) - \cos\phi_s\right)$$

Constant of Motion (Longitudinal "Hamiltonian")

$$\frac{d\Delta\phi}{dt}\frac{d^2\Delta\phi}{dt^2} = -\frac{2\pi\eta_c}{\lambda p T_0} eV_c \frac{d\Delta\phi}{dt}\left(\cos\left(\phi_s + \Delta\phi\right) - \cos\phi_s\right)$$

$$\frac{1}{2}\left(\frac{d\Delta\phi}{dt}\right)^2 = -\frac{2\pi\eta_c}{\lambda p T_0} eV_c \left(\sin\left(\phi_s + \Delta\phi\right) - \Delta\phi\cos\phi_s\right) + C$$

$$H\left(\Delta\phi, T_0\Delta E\right) = \frac{1}{2}\frac{2\pi\eta_c}{\lambda p T_0}\left(T_0\Delta E\right)^2 + eV_c\left(\sin\left(\phi_s + \Delta\phi\right) - \Delta\phi\cos\phi_s\right)$$

Equations of Motion

If neglect the slow (adiabatic) variation of p and T_0 with time, the equations of motion approximately Hamiltonian

$$\frac{d\Delta\phi}{dt} = \frac{\partial H}{\partial(T_0 \Delta E)} \qquad \frac{d(T_0 \Delta E)}{dt} = -\frac{\partial H}{\partial \Delta\phi}$$

In particular, the Hamiltonian is a constant of the motion

Kinetic Energy Term

$$T = \frac{1}{2}\frac{2\pi\eta_c T_0}{\lambda p}(\Delta E)^2$$

Potential Energy Term

$$V = eV_c\left(\sin(\phi_s + \Delta\phi) - \Delta\phi\cos\phi_s\right)$$

No Acceleration

$$\phi_s = \pm\pi/2 \qquad V = eV_c \cos\Delta\phi$$

$$\frac{d^2\Delta\phi}{dt^2} = \omega_s^2 \sin\Delta\phi$$

Better known as the real pendulum.

With Acceleration

$$\frac{d^2\Delta\phi}{dt^2} = \frac{\omega_s^2}{\sin\phi_s}\left(\cos\left(\phi_s + \Delta\phi\right) - \cos\phi_s\right)$$

$$\frac{1}{2}\left(\frac{d\Delta\phi}{dt}\right)^2 = \frac{\omega_s^2}{\sin\phi_s}\left(\sin\left(\phi_s + \Delta\phi\right) - \Delta\phi\cos\phi_s\right) + C$$

Equation for separatrix yields "fish" diagrams in phase space.
Fixed points at

$$\cos\left(\phi_s + \Delta\phi\right) = \cos\phi_s \qquad \Delta\phi = 0, -2\phi_s$$