Physics 696
Topics in Advanced Accelerator Design I

G. A. Krafft
Jefferson Lab
Jefferson Lab Professor of Physics
Old Dominion University

Relativity Theory

- Two Fundamental Postulates
- *Principle of Relativity*: All laws of nature (equations of physics) are identical in all inertial reference systems. Equations expressing laws of nature are invariant with respect to transformations of coordinates and time from one system to another. The equation describing any law of nature, when written in terms of coordinates and time in different inertial systems, has identical form in the different inertial systems. (Statement of Landau and Lifshitz, *Classical Theory of Fields*)
- *Invariance of Velocity of Light*: The speed of light is finite and independent of the velocity of the source or observer. (Jackson, *Classical Electrodynamics*)

Lorentz Transformations

- Maxwell Equations exhibit this symmetry. By postulate 2, all Laws of Physics must be of form to guarantee the invariance of the space-time interval

$$(ct')^2 - x'^2 - y'^2 - z'^2 = (ct)^2 - x^2 - y^2 - z^2$$

- Coordinate transformations that leave interval unchanged are the usual rotations and Lorentz transformations, e.g. the z boost

$$ct' = \gamma(ct - \beta z)$$

$$x' = x$$

$$y' = y$$

$$z' = \gamma(z - \beta ct)$$

Relativistic Factors

- Where, following Einstein define the relativistic factors

$$\vec{\beta} = \frac{\vec{\mathrm{v}}}{c} \qquad \beta = \frac{|\vec{\mathrm{v}}|}{c}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

- Easy way to accomplish task of defining a *Relativistic Mechanics*: write all laws of physics in terms of 4-vectors and 4-tensors, i.e., quantities that transform under Lorentz transformations in the same way as the coordinate differentials.

Four-vectors

- Four-vector transformation under *z* boost Lorentz Transformation

$$v^{0\prime} = \gamma\left(v^0 - \beta v^3\right)$$

$$v^{1\prime} = v^1$$

$$v^{2\prime} = v^2$$

$$v^{3\prime} = \gamma\left(v^3 - \beta v^0\right)$$

- Important example: Four-velocity. Note that interval

$$d\tau \equiv \sqrt{1-\beta^2}\,dt$$

Lorentz invariant. So the following is a 4-vector

$$cu^\alpha \equiv \left(\frac{dct}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right) = c\gamma\left(1, \beta_x, \beta_y, \beta_z\right)$$

4-Momentum

- Single particle mechanics must be defined in terms of Four-momentum

$$p^\alpha \equiv mcu^\alpha = mc\gamma\left(1, \beta_x, \beta_y, \beta_z\right)$$

- Norms, which must be Lorentz invariant, are

$$\sqrt{u_\alpha u^\alpha} \equiv 1, \sqrt{p_\alpha p^\alpha} \equiv mc \rightarrow E^2 = p^2c^2 + m^2c^4$$

- What happens to Newton's Law $\vec{F} = m\vec{a} = d\vec{p} / dt$?

$$\frac{dp^\alpha}{d\tau} \equiv F^\alpha$$

- But need a Four-force on the RHS!!!

Electromagnetic (Lorentz Force)

Non-relativistic

$$\vec{F} = q\left(\vec{E} + \vec{\mathrm{v}} \times \vec{B}\right)$$

Relativistic generalization (ν summation implied)

$$F^{\alpha} = qF^{\alpha}{}_{\nu}u^{\nu}$$

Matching with non-relativistic limit, electromagnetic field tensor must be

$$F^{\alpha}{}_{\nu} \equiv \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{pmatrix}$$

Relativistic Mechanics in E-M Field

- Energy Exchange Equation (Note: no magnetic field! Will show later)

$$\frac{d\gamma}{dt} = \frac{q\vec{E}\cdot\vec{\mathrm{v}}}{mc^2}$$

- Relativistic Lorentz Force Equation (you verify in HW!)

$$\frac{d\left(\gamma m\vec{\mathrm{v}}\right)}{dt} = q\left(\vec{E} + \vec{\mathrm{v}}\times\vec{B}\right)$$

- Conclusion: to get the effect of EM field on a relativistic particle, replace

$$m \rightarrow \gamma m$$

Relativistic Lagrangian

- Action integral for free particle must generate correct equations of motion for all possible Lorentz frames where the calculation done. Therefore, must a Lorentz scalar, or some function of Lorentz scaler. For free particle, only scalar available is the proper time

$$S = \alpha \int_{\tau_1}^{\tau_2} d\tau = \alpha \int_{t_1}^{t_2} \sqrt{1-\beta^2}\, dt$$

- Taking non-relavistic limit $\beta << 1$, and comparing to the non-relativistic Lagrangian gives

$$\alpha\sqrt{1-\beta^2} \rightarrow -\frac{\alpha\beta^2}{2} = \frac{m\mathrm{v}^2}{2} \qquad \therefore \alpha = -mc^2$$

$$L_{\text{free particle}} = -mc^2\sqrt{1-\beta^2}$$

Free Particle Hamiltonian

- Free particle canonical momentum

$$\vec{p} = \gamma m c \vec{\beta} = \gamma m \vec{\mathrm{v}}$$

- Free particle hamiltonian (total energy)

$$H\left(\vec{p}\right) = \vec{p} \cdot \frac{\vec{p}}{\gamma m} + \frac{mc^2}{\gamma} = \frac{mc^2}{\gamma}\left(\beta^2\gamma^2 + 1\right)$$

$$H\left(\vec{p}\right) = \frac{mc^2}{\sqrt{1 - p^2 / \left(\gamma m c\right)^2}} = \gamma m c^2$$

- Relativistic mass-energy relation

$$H^2 = p^2 c^2 + m^2 c^4$$

- Rest energy, when momentum vanishes

$$E_{rest} = mc^2$$

With EM Forces

- When include EM forces, γL, must be a Lorentz scalar as for a free particle. Only possibility linear in the EM 4-potential $A^\mu=(\varphi/c,A_x,A_y,A_z)$ is

$$\gamma L = -mc^2 - qu_\mu A^\mu \rightarrow L = -mc^2\sqrt{1-\beta^2} - q\phi + q\vec{\mathrm{v}}\cdot\vec{A}$$

  The sign of the added term is chosen so that the electromagnetic forces follow the usual sign conventions
- Already seen this addition to the Lagrangian gives the Lorentz Force, and therefore the relativistic equations of motion
- Canonical Momentum

$$\vec{P} = \gamma mc\vec{\beta} + q\vec{A} = \gamma m\vec{\mathrm{v}} + q\vec{A} = \vec{p} + q\vec{A}$$

Hamiltonian

- Follow the usual prescription

$$H = \vec{P} \cdot \dot{\vec{x}} - L = \vec{P} \cdot \left(\frac{\vec{P} - q\vec{A}}{\gamma m} \right) + \frac{mc^2}{\gamma} + q\phi - q\left(\frac{\vec{P} - q\vec{A}}{\gamma m} \right) \cdot \vec{A}$$

$$= \frac{\left(\vec{P} - q\vec{A}\right) \cdot \left(\vec{P} - q\vec{A}\right)}{\gamma m} + \frac{mc^2}{\gamma} + q\phi$$

$$\therefore H - q\phi, \vec{P} - q\vec{A}$$

solve the free particle mass-energy relation (also called minimal coupling prescription)

$$\left(H - q\phi\right)^2 = \left(\vec{P} - q\vec{A}\right)^2 c^2 + m^2 c^4$$

$$H = \sqrt{\left(\vec{P} - q\vec{A}\right)^2 c^2 + m^2 c^4} + q\phi$$

Energy Exchange Equation

- H is "conserved" ($dH/dt = 0$) when no explicit time dependence in Lagrangian ($dH/dt = \partial H/\partial t = -\partial L/\partial t$ when there is explicit time dependence). Total energy of the particle as it moves in the field is $H - q\varphi = \gamma mc^2$

$$\frac{d\gamma mc^2}{dt} = \frac{d}{dt}\sqrt{p^2c^2 + m^2c^4} = \frac{2\vec{p}\cdot\dot{\vec{p}}c^2}{2\sqrt{p^2c^2 + m^2c^4}}$$

$$= \vec{\mathrm{v}}\cdot\frac{d\vec{p}}{dt} = q\vec{\mathrm{v}}\cdot\left[\vec{E} + \vec{\mathrm{v}}\times\vec{B}\right] = q\vec{\mathrm{v}}\cdot\vec{E}$$

- This justifies the first row in the expression for the relativistic force law for the electromagnetic field
- Therefore, note that

$$\dot{\vec{\mathrm{v}}} = \frac{q}{\gamma m}\left\{\vec{E} + \vec{\mathrm{v}}\times\vec{B} - \vec{\mathrm{v}}\left(\vec{\mathrm{v}}\cdot\vec{E}\right)/c^2\right\}$$

Transformation of EM Field

- Occasionally it is easier to perform a calculation in a moving frame of reference stationary with the beam. In this instance, need to have EM field in the moving frame
- EM field is a 2-tensor of form

$$F^{uv} \equiv \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}$$

$$ct' = \gamma\left(ct - \vec{\beta}\cdot\vec{x}\right)$$

$$\vec{x}' = \vec{x} + \frac{(\gamma-1)}{\beta^2}\left(\vec{\beta}\cdot\vec{x}\right)\vec{\beta} - \gamma\vec{\beta}ct$$

$$F'^{u\nu} \equiv \frac{\partial x'^{\mu}}{\partial x^{\eta}} \frac{\partial x'^{\nu}}{\partial x^{\iota}} F^{\eta\iota} \qquad F^{\eta\iota} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}$$

$$\vec{E}' = \gamma\left(\vec{E} + \vec{\mathrm{v}} \times \vec{B}\right) - \frac{\gamma^2}{\gamma+1}\vec{\beta}\left(\vec{\beta}\cdot\vec{E}\right)$$

$$\vec{B}' = \gamma\left(\vec{B} - \vec{\mathrm{v}} \times \vec{E} / c^2\right) - \frac{\gamma^2}{\gamma+1}\vec{\beta}\left(\vec{\beta}\cdot\vec{B}\right)$$

Lorentz Group

- Linear coordinate transformations leaving norm invariant

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} L^0_0 & L^0_1 & L^0_2 & L^0_3 \\ L^1_0 & L^1_1 & L^1_2 & L^1_3 \\ L^2_0 & L^2_1 & L^2_2 & L^2_3 \\ L^3_0 & L^3_1 & L^3_2 & L^3_3 \end{pmatrix} \begin{pmatrix} ct = x^0 \\ x = x^1 \\ y = x^2 \\ z = x^3 \end{pmatrix}$$

$$x'_\mu x'^\mu = x_\mu x^\mu \rightarrow x'^\mu g_{\mu\nu} x'^\nu = x^\mu g_{\mu\nu} x^\nu$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Generators

$$x'^{\mu} g_{\mu\nu} x'^{\nu} = \begin{pmatrix} x'^0 & x'^1 & x'^2 & x'^3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix}$$

$$= \begin{pmatrix} x^0 & x^1 & x^2 & x^3 \end{pmatrix} \begin{pmatrix} L^0_0 & L^1_0 & L^2_0 & L^3_0 \\ L^0_1 & L^1_1 & L^2_1 & L^3_2 \\ L^0_2 & L^1_2 & L^2_2 & L^3_2 \\ L^0_3 & L^1_3 & L^2_3 & L^3_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} L^0_0 & L^0_1 & L^0_2 & L^0_3 \\ L^1_0 & L^1_1 & L^1_2 & L^1_3 \\ L^2_0 & L^2_1 & L^2_2 & L^2_3 \\ L^3_0 & L^3_1 & L^3_2 & L^3_3 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$= (Lx)^t gLx = x^t \left(L^t g L \right) x \quad \therefore g = L^t g L \qquad (I+\varepsilon)^t g (I+\varepsilon) = g$$

$$\varepsilon^t g = \varepsilon^t g^t = (g\varepsilon)^t = -g\varepsilon \rightarrow \varepsilon = \begin{pmatrix} 0 & \alpha & \beta & \gamma \\ \alpha & 0 & \delta & -\eta \\ \beta & -\delta & 0 & \iota \\ \gamma & \eta & -\iota & 0 \end{pmatrix}$$

Rotations/Boosts

$$R_z(\theta) = e^{\theta L_z} \qquad L_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad L_z^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$L_z^3 = -L_z \to R_z(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B_x(\xi) = e^{\xi G_x} \qquad G_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad G_x^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$G_x^3 = G_x \to B_x(\xi) = \begin{pmatrix} \cosh\xi & \sinh\xi & 0 & 0 \\ \sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \tanh\xi = \beta \quad \text{rapidity adds}$$

Link with E-M Field

$$F^{\alpha}{}_{\nu} \equiv \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{pmatrix}$$

For constant $F^{\alpha}{}_{\nu}$

$$\frac{du^{\alpha}}{d\tau} = \frac{qF^{\alpha}{}_{\nu}u^{\nu}}{mc} \rightarrow \begin{pmatrix} u^0(\tau) \\ u^1(\tau) \\ u^2(\tau) \\ u^3(\tau) \end{pmatrix} = \exp\left(\frac{qF}{mc}\tau\right)\begin{pmatrix} u^0(\tau=0) \\ u^1(\tau=0) \\ u^2(\tau=0) \\ u^3(\tau=0) \end{pmatrix}$$

- (E_x, E_y, E_z) generate dynamically boosts in the (x,y,z) directions respectively
- (B_x, B_y, B_z) generate dynamically rotations around the (x,y,z) axes respectively