Physics 696
Topics in Advanced Accelerator Design I

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Mechanics

In accelerators, the principles of **Mechanics** are used to determine and solve the equations of motion of the particles that make up the beam in the accelerator. Depending on the energy of the particles involved, one may need to use either non-relativistic (Newtonian) mechanics or the relativistic mechanics of Einstein and Lorentz. Non-relativistic mechanics starts with

$$\vec{F} = m\vec{a} = \frac{d\vec{p}}{dt} \qquad \vec{p} = m\vec{\mathrm{v}} \text{ momentum (2nd Law)}$$

$$\vec{F} = 0 \rightarrow \vec{p} = \text{constant} \quad \text{momentum conserved (1st Law)}$$

$$\vec{\tau} = \vec{r} \times \vec{F} = 0 \rightarrow \vec{L} = \vec{r} \times \vec{p} = \text{constant}$$

angular momentum conserved

Work/Energy

- Work done by an external force

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{s} = \int_1^2 \frac{d}{dt}\left(m\vec{\mathrm{v}}\right) \cdot d\vec{s} \qquad \text{Work moving from 1 to 2}$$

$$= \int_1^2 \frac{d}{dt}\left(\frac{m\vec{\mathrm{v}} \cdot \vec{\mathrm{v}}}{2}\right) dt = \frac{m\mathrm{v}_2^2}{2} - \frac{m\mathrm{v}_1^2}{2}$$

- "Conservative" force has Energy Conservation

$$\vec{\nabla} \times \vec{F} = 0 \rightarrow \vec{F} = -\vec{\nabla} V \qquad \exists \text{ Potential function}$$

$$W_{12} = -\int_1^2 \nabla V \cdot d\vec{s} = -\int_1^2 dV = V_1 - V_2 \quad \text{path independent}$$

$$\frac{m\mathrm{v}^2}{2}\left(\text{Kinetic Energy, } T\right) + V\left(\vec{x}\right)\left(\text{Potential Energy}\right) = \text{constant}$$

Potential Function Examples

- Gravitational potential near surface of earth

$$mgz$$

- Gravitational potential surrounding a star/planet

$$G\frac{m_s m}{r}$$

- Potential for a static electric charge

$$\frac{1}{4\pi\varepsilon_0}\frac{q_1 q}{r}$$

Action

- Newton's 2nd Law can be difficult to apply in non-Cartesian coordinate systems (e.g., cylindrical or spherical coordinate systems)
- Find a procedure equivalent to Newton's 2nd Law in Cartesian system, but can be used more generally
- Start by defining the "Lagrangian"

$$L = \frac{m\mathrm{v}^2}{2} - V\left(\vec{x}\right) \qquad \vec{F}_{cart} = \frac{d}{dt}\left[\frac{\partial L}{\partial \vec{\mathrm{v}}}\right] = \frac{\partial L}{\partial \vec{x}}$$

- Action is the integral of the Lagrangian along the motion

$$S = \int_{t_1}^{t_2} L\left(\vec{\mathrm{v}}\left(t\right), \vec{x}\left(t\right)\right) dt$$

Principle of Extremal (Least) Action

- Actual motion of the particle (equations of motion in Lagrangian form)

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \vec{\mathrm{v}}}\right]-\frac{\partial L}{\partial \vec{x}}=0$$

makes the action integral extremal. Work the argument the other way. In a general coordinate system, the equations of motion must be of the same Euler-Lagrange form.

$$\delta S=\int_{t_1}^{t_2}\left[\frac{\partial L}{\partial \vec{\mathrm{v}}}\cdot\delta\vec{\mathrm{v}}(t)+\frac{\partial L}{\partial \vec{x}}\cdot\delta\vec{x}(t)\right]dt=\int_{t_1}^{t_2}\left[\frac{\partial L}{\partial \vec{\mathrm{v}}}\cdot\frac{d}{dt}\delta\vec{x}(t)+\frac{\partial L}{\partial \vec{x}}\cdot\delta\vec{x}(t)\right]dt$$

$$=\int_{t_1}^{t_2}\left[-\frac{d}{dt}\frac{\partial L}{\partial \vec{\mathrm{v}}}+\frac{\partial L}{\partial \vec{x}}\right]\cdot\delta\vec{x}(t)\,dt \qquad \delta S=0\Leftrightarrow\frac{d}{dt}\frac{\partial L}{\partial \vec{\mathrm{v}}}-\frac{\partial L}{\partial \vec{x}}=0$$

Example Cylindrical Coordinates

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\dot{\theta}$$

$$\dot{y} = \dot{r}\sin\theta + r\cos\theta\dot{\theta}$$

$$L = \frac{m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)}{2} - V\left(r,\theta\right)$$

$$\frac{d}{dt}\left(m\dot{r}\right) - r\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$$

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) + \frac{r}{r}\frac{\partial V}{\partial \theta} = 0 = r\left[mr\ddot{\theta} + 2\dot{r}\dot{\theta} - F_\theta\right]$$

Canonical or Conjugate Momentum

- For general Lagrangian, the canonical momentum is

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

- Cylindrical Coordinates

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

Do not always have momentum units

- For cyclic coordinates

$$\frac{\partial L}{\partial q_i} = 0 \rightarrow \frac{dp_i}{dt} = 0 \rightarrow p_i = \text{constant}$$

canonical momentum is conserved. For cylindrical systems

$$p_\theta = \text{constant} \qquad \text{if } V\left(r,\theta\right) = V\left(r\right)$$

Hamiltonian (Energy) Function

- Euler-Lagrange equations are second order ordinary differential equations.
- In some situations it is easier to replace each Euler-Lagrange equation with *two* first order ODEs, treating (canonical) momentum and the coodinates on the same footing. Key is finding a function of the coordinates and momenta, that corresponds to the total energy (the Hamiltonian)

$$\frac{dL}{dt} = \sum_i \left[\frac{\partial L}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt} + \frac{\partial L}{\partial q_i} \frac{dq_i}{dt} \right] = \sum_i \left[p_i \frac{d\dot{q}_i}{dt} + \frac{dp_i}{dt} \frac{dq_i}{dt} \right]$$

$$\frac{d}{dt}\left[L - \sum_i p_i \frac{dq_i}{dt} \right] = 0 \rightarrow -H = L - \sum_i p_i \frac{dq_i}{dt}$$

Legendre Transformation

1. Solve for generalized velocities in terms of momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \rightarrow \dot{q}_i = \dot{q}_i \left(q_i, p_j \right)$$

2. Replace the generalized velocities by momenta

$$L\left(q_i, \dot{q}_j \right) \rightarrow L\left(q_i, \dot{q}_j \left(q_i, p_j \right) \right)$$

3. Determine the Hamiltonian (energy) function in terms of coordinates and momenta

$$H\left(q_i, p_j \right) \rightarrow \sum_k \dot{q}_k \left(q_i, p_j \right) p_k - L\left(q_i, \dot{q}_j \left(q_i, p_j \right) \right)$$

Hamilton's Canonical Equations

- For kinetic energy quadratic in the $\dot{q}_i$

$$H = \sum_i \dot{q}_i p_i - L = T + V$$

- Treat coordinates and conjugate momenta on same footing

$$dH = \sum_i \left[\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right] + \frac{\partial H}{\partial t} dt$$

$$= \sum_i \dot{q}_i dp_i + \sum_i p_i d\dot{q}_i - \sum_i \left[\frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \right] - \frac{\partial L}{\partial t} dt$$

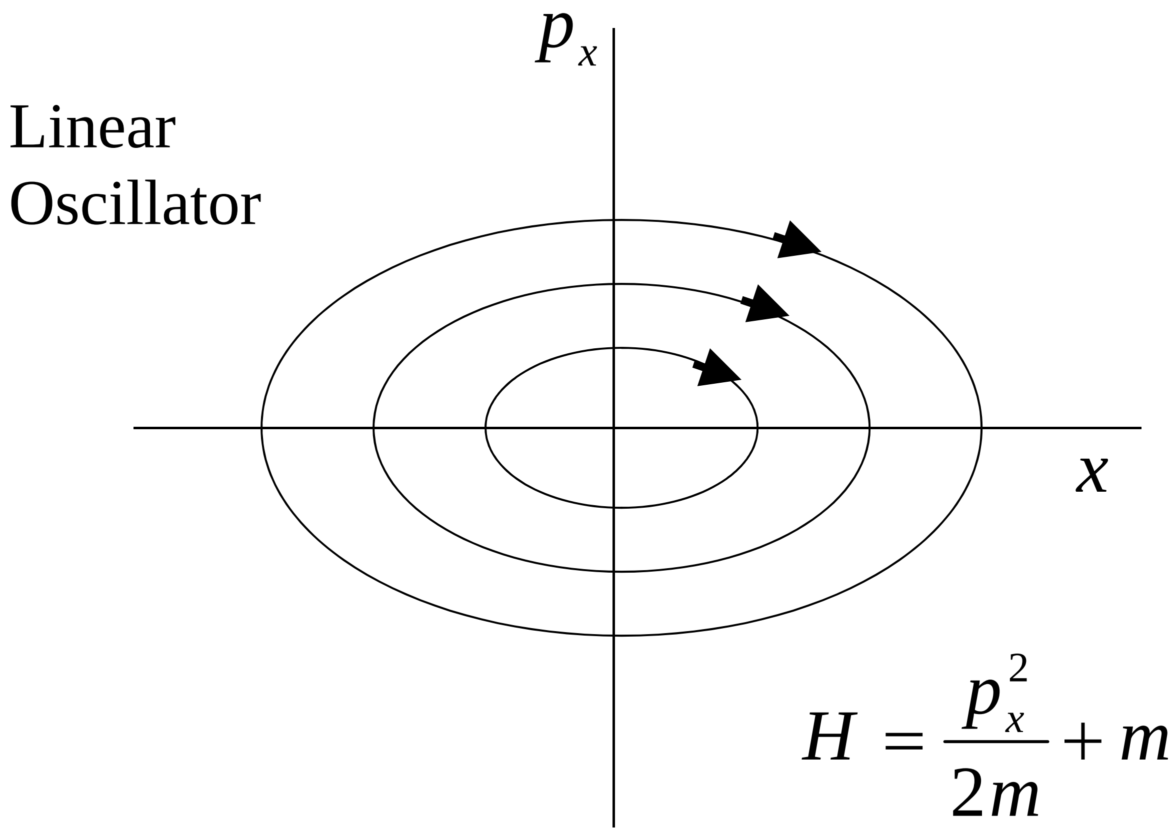
$$\frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} = -\frac{dp_i}{dt} \qquad \frac{\partial H}{\partial p_i} = \frac{dq_i}{dt} \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

Canonical Transformations

- Hamiltonian outlook has coordinates and momenta on same footing, and various transformations between them are allowed
- There is a large body of theory, called canonical transformation theory, dealing with problem of which transformations of coordinates and momenta leave Hamilton's equations form invariant. Transformations that do are called canonical transformations
- The fact that choosing what quantities in the theory to be coordinates and what quantities in the theory are momenta is arbitrary through canonical transformation, encourages one to picture dynamic motion as movement through "phase space"

Phase Space

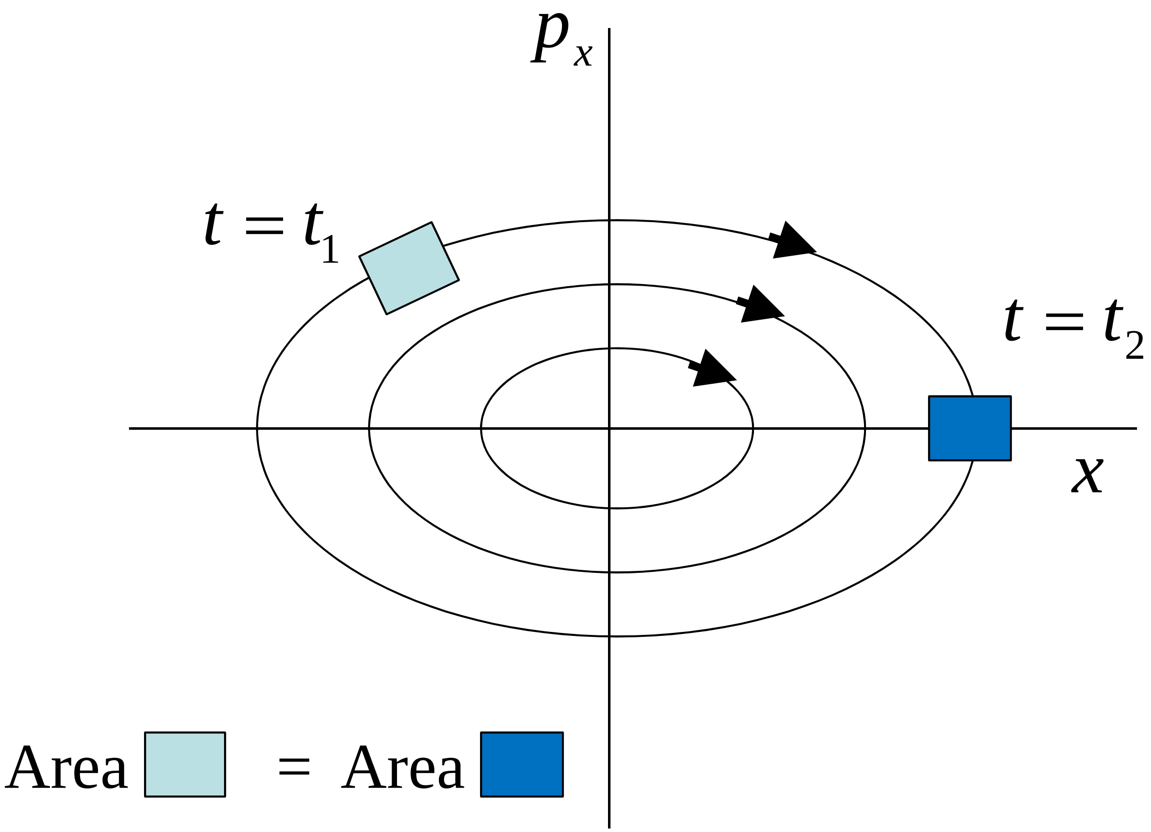
- Plot of dynamical system "state" with coordinate along abscissa and momentum along the ordinate



$$H = \frac{p_x^2}{2m} + m\omega^2 \frac{x^2}{2}$$

Liouville Theorem

- Area in phase space is preserved when the dynamics is Hamiltonian



1D Proof

$$\frac{dV}{dt} = \lim_{\Delta t \to \infty} \frac{V(t+\Delta t) - V(t)}{\Delta t}$$

$$x(s, t+\Delta t) \doteq x(s) + \left.\frac{dx}{dt}\right|_t \Delta t + \cdots \doteq x(s) + \frac{\partial H}{\partial p_x}\big(x(s), p_x(s)\big)\Delta t + \cdots$$

$$p_x(s, t+\Delta t) \doteq p_x(s) + \left.\frac{dp_x}{dt}\right|_t \Delta t + \cdots \doteq p_x(s) - \frac{\partial H}{\partial x}\big(x(s), p_x(s)\big)\Delta t + \cdots$$

$$V(t) = \oint_{c(s,t)} p_x dx = \int_0^L p_x(s,t) \frac{dx}{ds}(s,t) ds$$

$$V(t+\Delta t) = \oint_{c(s,t+\Delta t)} p_x dx = \int_0^L p_x(s,t+\Delta t) \frac{dx}{ds}(s,t+\Delta t) ds$$

$$\doteq \int_0^L \left[p_x(s) + \frac{\partial H}{\partial x}\big(x(s), p_x(s)\big)\Delta t \right] \frac{d}{ds}\left[x(s) - \frac{\partial H}{\partial p_x}\big(x(s), p_x(s)\big)\Delta t \right] ds$$

$$\frac{dV}{dt} = \int_0^L \left[-p_x(s) \frac{d}{ds} \frac{\partial H}{\partial p_x}\left(x(s), p_x(s)\right) + \frac{\partial H}{\partial x}\left(x(s), p_x(s)\right) \frac{dx(s)}{ds} \right] ds$$

$$= \int_0^L \left[\frac{dp_x(s)}{ds} \frac{\partial H}{\partial p_x}\left(x(s), p_x(s)\right) + \frac{\partial H}{\partial x}\left(x(s), p_x(s)\right) \frac{dx(s)}{ds} \right] ds$$

(why is boundary term of integration by parts zero?)

$$= \oint_{c(s,t)} \left[\frac{\partial H}{\partial p_x} dp_x + \frac{\partial H}{\partial x} dx \right]$$

By Green's Thm.

=0 when the differential is an exact differential

i.e., $\frac{\partial}{\partial x}\left(\frac{\partial H}{\partial p_x}\right) = \frac{\partial}{\partial p_x}\left(\frac{\partial H}{\partial x}\right)$, in other words always

(note the integrand above is really dH, so H is a "potential" for phase space!!!)

3D Poincare Invariants

- In a three dimensional Hamiltonian motion, the 6D phase space volume is conserved (also called Liouville's Thm.)

$$V_{6D} = \int_{V_6} dp_x dp_y dp_z dx dy dz$$

- Additionally, the sum of the projected volumes (Poincare invariants) are conserved

$$\int_{\mathrm{proj}(V_2)} dp_x dx + \int_{\mathrm{proj}(V_2)} dp_y dy + \int_{\mathrm{proj}(V_2)} dp_z dz$$

$$\int_{\mathrm{proj}(V_4)} dp_y dp_z dy dz + \int_{\mathrm{proj}(V_4)} dp_z dp_x dz dx + \int_{\mathrm{proj}(V_4)} dp_x dp_y dx dy$$

Emittance (phase space area) exchange based on this idea

- More complicated to prove, but are true because, as in 1D

$$\frac{\partial^2 H}{\partial q_i \partial p_i} = \frac{\partial^2 H}{\partial p_i \partial q_i}$$

γ is a loop in 6D phase space

$$\gamma(t) = \left(\vec{p}(s,t), \vec{q}(s,t)\right)$$

$$\frac{d}{dt}\left[\oint_{\gamma(t)} \sum_{i=1}^{3} p_i dx_i\right] = \int_0^L \sum_{i=1}^{3}\left[-p_i(s)\frac{d}{ds}\frac{\partial H}{\partial p_i}\left(\vec{x}(s), \vec{p}(s)\right) + \frac{\partial H}{\partial x_i}\left(\vec{x}(s), \vec{p}(s)\right)\frac{dx_i(s)}{ds}\right]$$

$$= \int_0^L \sum_{i=1}^{3}\left[\frac{dp_i(s)}{ds}\frac{\partial H}{\partial p_i}\left(\vec{x}(s), \vec{p}(s)\right) + \frac{\partial H}{\partial x_i}\left(\vec{x}(s), \vec{p}(s)\right)\frac{dx_i(s)}{ds}\right] = \oint_{\gamma(t)} dH = 0$$

for any surface in 6D phase space V_2, with $\gamma = \partial V_2$

$$\oint_{\partial V_2} \sum_{i=1}^{3} p_i dx_i = \int_{V_2} \sum_{i=1}^{3} dp_i dx_i = \sum_{i=1}^{3} \int_{\mathrm{proj}(V_2)} dp_i dx_i$$

$$\left(\sum_{i=1}^{3} dp_i dx_i\right)^2 = dp_y dp_z dy dz + dp_z dp_x dz dx + dp_x dp_y dx dy$$

$$\left(\sum_{i=1}^{3} dp_i dx_i\right)^3 = dp_x dp_y dp_z dx dy dz$$

Electromagnetic Force

- Lorentz Force

$$\vec{F} = q\left(\vec{E} + \vec{\mathrm{v}} \times \vec{B}\right)$$

- Equations of motion follow from a "velocity-dependent" Lagrangian (non-relativistic case) (Verify in HW)

$$L = m\frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{2} - q\phi\left(x, y, z\right) + q\left(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z\right)$$

- Canonical momentum

$$P_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA_x \quad P_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + qA_y \quad P_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} + qA_z$$

Hamiltonian for Particle in EM Field

- Hamiltonian is

$$
\begin{aligned}
H &= \sum_i P_i \dot{x}_i - L \\
&= \vec{P} \cdot \left(\frac{\vec{P} - q\vec{A}}{m} \right) - \frac{m}{2} \left(\frac{\vec{P} - q\vec{A}}{m} \right) \cdot \left(\frac{\vec{P} - q\vec{A}}{m} \right) + q\phi - q \left(\frac{\vec{P} - q\vec{A}}{m} \right) \cdot \vec{A} \\
&= \frac{\vec{P} \cdot \vec{P}}{2m} - \frac{q}{m} \vec{P} \cdot \vec{A} + \frac{q}{m} \vec{P} \cdot \vec{A} - \frac{q}{m} \vec{P} \cdot \vec{A} + \frac{q^2}{2m} \vec{A} \cdot \vec{A} + q\phi \\
&= \frac{\left(\vec{P} - q\vec{A}\right) \cdot \left(\vec{P} - q\vec{A}\right)}{2m} + q\phi \qquad \therefore \text{ still } T + V
\end{aligned}
$$

- Equations of motion gauge invariant

$$
\vec{A}' = \vec{A} + \vec{\nabla}\lambda \qquad \phi' = \phi - \partial\lambda / \partial t
$$

$$
L' = \frac{m\dot{\vec{x}} \cdot \dot{\vec{x}}}{2} - q\phi' + q\dot{\vec{x}} \cdot \vec{A}' = L + q\frac{d\lambda}{dt}
$$

- Verify Hamilton's equations yield the Lorentz Force

Liouville’s Theorem Again

- From Liouville’s theorem we can conclude right away that, neglecting interactions between particles, the phase space density in a particle beam cannot be changed merely by acting on the beam with an external Electromagnetic field. Standard magnetic field configurations will not change particle phase space density. Important point, and a fundamental problem in accelerator physics. Beam quality usually tends to degrade as one propagates along the accelerator.