TAAD1
Electromagnetic Theory

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Classical Electrodynamics

- A main physics discovery of the last half of the 20th century is that electrodynamics is a part of an overarching quantum theory of nature called the *standard model*
- In the standard model, electromagnetism is closely linked with the weak interaction force, through the so-called electroweak unification, in a non-Abelian gauge theory that relies on and utilizes the usual phase invariance of quantum mechanics
- As part of the standard model, electromagnetic forces are distinguished in that they couple to electric charge, and they are long (∞ ?) range because to the best of present knowledge, the photon (EM force carrier) is (nearly?) massless
- This fact "explains" their prevalence in our day-to-day life

Range of Validity of Classical Description

- Large numbers of quanta (quantum mechanical fluctuations are relatively small, leading to classical results)

$$N_{photons} >> 1$$

- Low photon energy

$$E_{photons} = \hbar\omega_{photons} << m_{electron}c^2$$

Higher energies can lead to (quantum mechanical) effects like Compton Effect. Not sufficient (photoelectric effect!)

- Macroscopic charge, i. e., large numbers of individual electrons. Electron discreteness not important.
- Does not work for atoms and new nano-scale devices. Need Quantum Mechanics for them!

Line and Surface Integrals

- Line integral

$$\int_L \vec{V} \cdot d\vec{l} \equiv \int_a^b \vec{V}\left(\vec{L}(\tau)\right) \cdot \frac{d\vec{L}(\tau)}{d\tau} d\tau$$

for *any* parameterization of L (invariant to choice!)

- Surface (Flux) integral

$$\int_S \vec{V} \cdot \vec{n} dA \equiv \iint_{S'} \vec{V}\left(\vec{S}(\sigma,\tau)\right) \cdot \left[\frac{\partial \vec{S}}{\partial \sigma} \times \frac{\partial \vec{S}}{\partial \tau}\right]\left(\vec{S}(\sigma,\tau)\right) d\sigma\, d\tau$$

$$\frac{\partial \vec{S}}{\partial \sigma}(\sigma,\tau) \equiv \lim_{\delta\sigma \to 0} \frac{\vec{S}(\sigma+\delta\sigma,\tau) - \vec{S}(\sigma,\tau)}{\delta\sigma}$$

again independent of parameters up to sign (Jacobian!).
Can usually integrate easily in high-symmetry situations.

Line and Surface Integrals

- Line integral parametrization invariant

$$\int_L \vec{V}\cdot d\vec{l} \equiv \int_a^b \vec{V}\left(\vec{L}\left(\tau(s)\right)\right)\cdot\frac{d\vec{L}\left(\tau(s)\right)}{ds}ds = \int_a^b \vec{V}\left(\vec{L}\left(\tau(s)\right)\right)\cdot\frac{d\vec{L}\left(\tau(s)\right)}{d\tau}\frac{d\tau}{ds}ds$$

$$= \int_{\tau(a)}^{\tau(b)} \vec{V}\left(\vec{L}(\tau)\right)\cdot\frac{d\vec{L}(\tau)}{d\tau}d\tau \qquad \text{by change of variables formula}$$

- Surface (Flux) integral parametrization invariant

$$\int_S \vec{V}\cdot\vec{n}dA \equiv \iint_{P'} \vec{V}\left(\vec{S}\left(\sigma(\sigma',\tau'),\tau(\sigma',\tau')\right)\right)\cdot\left[\frac{\partial\vec{S}}{\partial\sigma'}\times\frac{\partial\vec{S}}{\partial\tau'}\right]\left(\vec{S}\left(\sigma(\sigma',\tau'),\tau(\sigma',\tau')\right)\right)d\sigma' d\tau'$$

$$\left[\frac{\partial\vec{S}}{\partial\sigma'}\times\frac{\partial\vec{S}}{\partial\tau'}\right] = \left[\frac{\partial\vec{S}}{\partial\sigma}\frac{\partial\sigma}{\partial\sigma'}+\frac{\partial\vec{S}}{\partial\tau}\frac{\partial\tau}{\partial\sigma'}\right]\times\left[\frac{\partial\vec{S}}{\partial\sigma}\frac{\partial\sigma}{\partial\tau'}+\frac{\partial\vec{S}}{\partial\tau}\frac{\partial\tau}{\partial\tau'}\right] = \left[\frac{\partial\vec{S}}{\partial\sigma}\times\frac{\partial\vec{S}}{\partial\tau}\right]\begin{vmatrix}\partial\sigma/\partial\sigma' & \partial\tau/\partial\sigma' \\ \partial\sigma/\partial\tau' & \partial\tau/\partial\tau'\end{vmatrix}$$

$$\therefore \int_S \vec{V}\cdot\vec{n}dA \equiv \iint_P \vec{V}\left(\vec{S}(\sigma,\tau)\right)\cdot\left[\frac{\partial\vec{S}}{\partial\sigma}\times\frac{\partial\vec{S}}{\partial\tau}\right]\left(\vec{S}(\sigma,\tau)\right)d\sigma d\tau \quad \text{by 2-D change of variables}$$

- When evaluating flux integrals, normal must be chosen. You choose correct vector order in product to get normal.

Flux for a Radial Field

$$\vec{V}(x,y,z) = f(r)\hat{r}$$

$$r = \sqrt{x^2 + y^2 + z^2} \qquad \hat{r} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{r}$$

$$\Phi_{\vec{V}} = \int_S \vec{V}\cdot\vec{n}\,dA = \int_{S^+} \vec{V}\cdot\vec{n}\,dA + \int_{S^-} \vec{V}\cdot\vec{n}\,dA$$

$$\int_{S^+} \vec{V}\cdot\vec{n}\,dA = \int_{x^2+y^2\le a^2} f(a)\frac{a}{\sqrt{a^2 - x^2 - y^2}}\,dx\,dy = f(a)2\pi a^2$$

$$\int_{S^-} \vec{V}\cdot\vec{n}\,dA = f(a)2\pi a^2 \qquad \therefore \Phi_{\vec{V}} = f(a)4\pi a^2$$

Vector Calculus Theorems

- Stokes Theorem

$$\int_S \vec{\nabla} \times \vec{A} \cdot \vec{n} dA = \int_{L=\partial S} \vec{A} \cdot d\vec{l}$$

- Divergence Theorem

$$\int_V \vec{\nabla} \cdot \vec{A} dV = \int_{S=\partial V} \vec{A} \cdot \vec{n} dA$$

Electromagnetic Quantities

Charge [C], Charge Density [C/m^3]

$$q \qquad \rho\left(\vec{x},t\right)$$

Electric Field Vector [V/m]

$$\vec{E}\left(\vec{x},t\right)$$

Electric Displacement Vector [C/m^2]

$$\vec{D}\left(\vec{x},t\right)$$

Magnetic Field Vector [A/m]

$$\vec{H}\left(\vec{x},t\right)$$

Magnetic Induction Vector [V sec/m^2]

$$\vec{B}\left(\vec{x},t\right)$$

Maxwell's Equations

$\vec{\nabla} \cdot \vec{D} = \rho$	Electric charge generates electric displacement
$\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$	Faraday's Law
$\vec{\nabla} \cdot \vec{B} = 0$	No magnetic charges
$\vec{\nabla} \times \vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t}$	Magnetic field generated by real or displacement current density

Maxwell's Equations are linear in the source terms ρ and $\vec{J}$. In general will generate linear (partial) differential equations to solve. Superposition valid!

Constitutive Relations

- Relationships between electric quantities

$$\vec{D} = \varepsilon \vec{E}$$

- Relationships between magnetic quantities

$$\vec{B} = \mu \vec{H}$$

- For most of course assume purely linear relationship exists. As you go on, you'll find there are non-linear dependences and other complications
- Better approximation for static phenomena
- In vacuum

$$\vec{D} = \varepsilon_0 \vec{E} \qquad \vec{B} = \mu_0 \vec{H}$$

$$\varepsilon_0 = 8.85 \times 10^{-12} \frac{\mathrm{C}^2}{\mathrm{Nt\ m}^2} \qquad \mu_0 = 4\pi \times 10^{-7} \frac{\mathrm{Nt\ sec}^2}{\mathrm{C}^2}$$

Lorentz Force

- Force on a (quantized) charge q

$$\vec{F} = q\left(\vec{E} + \vec{\mathrm{v}} \times \vec{B}\right)$$

- From dimensions, see that force is coupled to magnetic induction, not magnetic field

- As will see later, valid at relativistic energies too when interpret force as change of relativistic momentum with time

$$\frac{d\left(\gamma m \vec{\mathrm{v}}\right)}{dt} = q\left(\vec{E} + \vec{\mathrm{v}} \times \vec{B}\right)$$

Maxwell Equations (Integral Form)

- Applying Divergence and Stokes Theorems

$$\vec{\nabla} \cdot \vec{D} = \rho \qquad \Rightarrow \oint_S \vec{D} \cdot \vec{n} \; dA = \int_V \rho \; dV$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \Rightarrow \oint_L \vec{E} \cdot d\vec{l} = -\oint_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{n} \; dA$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \Rightarrow \oint_S \vec{B} \cdot \vec{n} \; dA = 0$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad \Rightarrow \oint_L \vec{H} \cdot d\vec{l} - \oint_S \frac{\partial \vec{D}}{\partial t} \cdot \vec{n} \; dA = \oint_S \vec{J} \cdot \vec{n} \; dA$$

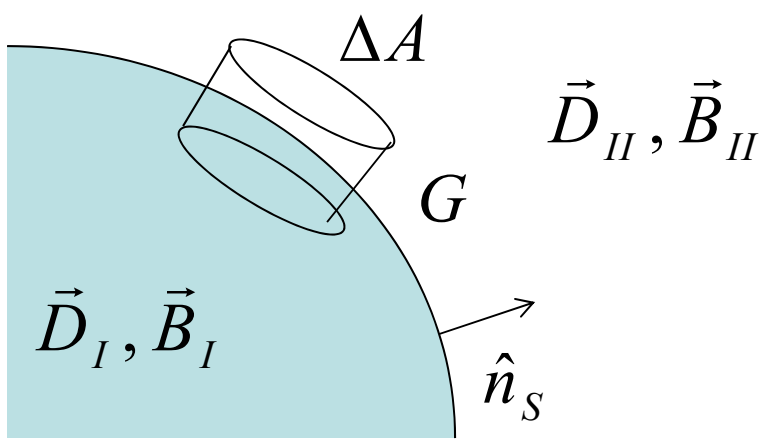
Boundary Conditions

- Gaussian Pillbox Argument, σ is surface charge

$$\oint_G \vec{D}\cdot\vec{n}\ dA \approx \left(\vec{D}_{II}-\vec{D}_I\right)\cdot\hat{n}_S\Delta A = \sigma\Delta A \rightarrow \left(\vec{D}_{II}-\vec{D}_I\right)\cdot\hat{n}_S = \sigma$$

$$\oint_G \vec{B}\cdot\vec{n}\ dA = \left(\vec{B}_{II}-\vec{B}_I\right)\cdot\hat{n}_S\Delta A = 0 \rightarrow \vec{B}_{II}\cdot\hat{n}_S - \vec{B}_I\cdot\hat{n}_S$$

- Discontinuity of normal component of D given by surface charge and no change in the normal component of B

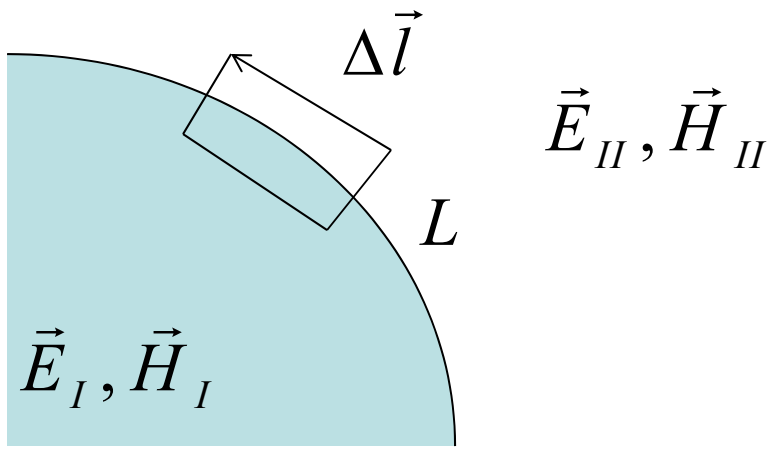


- Gaussian Loop Argument, K is surface current [A/m]

$$\oint_L \vec{E}\cdot d\vec{l} = -\oint_S \frac{\partial \vec{B}}{\partial t}\cdot \vec{n}\ dA \rightarrow \vec{E}_{II}\cdot d\vec{l} - \vec{E}_I \cdot d\vec{l} = 0$$

$$\oint_L \vec{H}\cdot d\vec{l} - \oint_S \frac{\partial \vec{D}}{\partial t}\cdot \vec{n}\ dA = \oint_S \vec{J}\cdot \vec{n}\ dA \rightarrow \vec{H}_{II}\cdot \Delta\vec{l} - \vec{H}_I \cdot \Delta\vec{l} - 0 = \vec{K}\cdot \hat{n}_S \times \Delta\vec{l}$$

- Discontinuity of tangential component of H given by surface current and no change in the tangential component of E



- Maxwell Equations plus the boundary conditions provide a complete description of all classical electromagnetic phenomena
- Given source functions and constitutive relations these equations have unique solutions
- A wide variety of numerical solutions exist; we want you to be able to understand the solutions that you derive from codes in your future working life!

Electrostatics

- Maxwell Equations simplify enormously if the fields and sources do not depend on time

- Electric field of a static charge q_1 at location $\vec{x}_1$ given by Coulomb's Law

$$\vec{E}\left(\vec{x}\right) = \frac{q_1}{4\pi\varepsilon_0}\frac{\vec{x}-\vec{x}_1}{\left|\vec{x}-\vec{x}_1\right|^3}$$

Coulomb Interaction

- Notation

$$\left|\vec{\mathrm{v}}\right| \equiv \sqrt{\mathrm{v}_x^2 + \mathrm{v}_y^2 + \mathrm{v}_z^2} \rightarrow$$

$$\left|\vec{x} - \vec{x}_1\right|^3 = \left(\left(x - x_1\right)^2 + \left(y - y_1\right)^2 + \left(z - z_1\right)^2\right)^{3/2}$$

- Force on test particle with charge q at location $\vec{x}$

$$\vec{F}\left(\vec{x}\right) = q\vec{E}\left(\vec{x}\right) = \frac{qq_1}{4\pi\varepsilon_0}\frac{\vec{x} - \vec{x}_1}{\left(\left(x - x_1\right)^2 + \left(y - y_1\right)^2 + \left(z - z_1\right)^2\right)^{3/2}}$$

- Dependence on the distance between charges

$$\left|\vec{F}\right| = \frac{qq_1}{4\pi\varepsilon_0}\frac{1}{\left(x - x_1\right)^2 + \left(y - y_1\right)^2 + \left(z - z_1\right)^2} \propto \frac{1}{r^2}$$

Gauss's Law

- Easy proof:

$$\vec{\nabla}\cdot\frac{\vec{x}-\vec{x}'}{\left|\vec{x}-\vec{x}'\right|^3}=\frac{1}{\left|\vec{x}-\vec{x}'\right|^3}-\frac{3\left(x-x'\right)^2}{\left|\vec{x}-\vec{x}'\right|^5}$$

$$+\frac{1}{\left|\vec{x}-\vec{x}'\right|^3}-\frac{3\left(y-y'\right)^2}{\left|\vec{x}-\vec{x}'\right|^5}+\frac{1}{\left|\vec{x}-\vec{x}'\right|^3}-\frac{3\left(z-z'\right)^2}{\left|\vec{x}-\vec{x}'\right|^5}=0 \quad \vec{x}\neq\vec{x}'$$

- Evaluate $\int\left(\vec{x}-\vec{x}'\right)/\left|\vec{x}-\vec{x}'\right|^3\cdot\vec{n}dA$ on a sphere surrounding $\vec{x}'$

$$\int_S\frac{\vec{x}-\vec{x}'}{\left|\vec{x}-\vec{x}'\right|^3}\cdot\vec{n}dA=\frac{1}{R^2}\int R^2 d\cos\theta d\phi=4\pi$$

- Independent of R! Gauss's Law follows from this!

Formal Statement

- Gauss's Law: for an *arbitrary* surface S (doesn't need to be a sphere!)

$$\int_S \vec{E} \cdot \vec{n} dA = \frac{q_{inside}}{\varepsilon_0}$$

- Pf: If a charge is outside, calculation 1 shows it doesn't contribute to the integral. If a charge is inside, calculation 2 shows it contributes to the integral as given.
- For any V and $S = \partial V$

$$\int_V \nabla \cdot \vec{E} dxdydz = \int_S \vec{E} \cdot \vec{n} dA = \frac{q_{inside}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_V \rho(\vec{x}) dxdydz$$

$$\therefore \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

1st Maxwell Eqn. in vacuum

Magnetostatics: Ampere's Law

In conventional vector notation is

$$\oint_L \vec{H} \cdot d\vec{l} = I$$

where I is the total current enclosed by the current loop.
Define current density vector, and current density

$$\vec{J} = \left(J_x, J_y, J_z \right)$$

For time-independent charge density

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} = 0.$$

Integrating over a surface bounded by L

$$\oint_{L=\partial S} \vec{H} \cdot d\vec{l} = \int_S \nabla \times \vec{H} \cdot \vec{n} dA = \int_S \vec{J} \cdot \vec{n} dA = I$$

Electromagnetic Potentials

Poincare's lemma plus Div $B = 0$ implies there is a vector A so that

$$\vec{B} = \nabla \times \vec{A}$$

This defines the usual vector potential. Poincare's lemma plus the Faraday's Law imply there is a scalar potential (sign chosen using traditional definition with E as the negative gradient)

$$\nabla \times \left[\vec{E} - \frac{\partial \vec{A}}{\partial t} \right] = 0 \rightarrow \vec{E} - \frac{\partial \vec{A}}{\partial t} = -\nabla \phi$$

Gauge Invariance

Considerable flexibility/latitude in choosing potentials. If redefine them by

$$A_{new} = A_{old} + d\Lambda$$

$$\phi_{new} = \phi_{old} - \frac{\partial \Lambda}{\partial t}$$

for an arbitrary space-time function, then the electric field and magnetic field will be identical when computed with the new potentials. In other words, the transformation from the old to new descriptions will leave the electromagnetic field invariant. One makes various choices on the gauge, for convenience of calculation.

Equations for the Potentials

$$\nabla \cdot \vec{D} = \rho \rightarrow \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho$$

$$\vec{D} = \varepsilon_0 \vec{E} \rightarrow \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] \phi + \frac{\partial}{\partial t} \left(\vec{\nabla} \cdot \vec{A} \right) = -\frac{\rho}{\varepsilon_0}$$

and, by sorting the individual components

$$\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}, \vec{H} = \frac{\vec{B}}{\mu_0} \rightarrow$$

$$\sum_{j,l,m=1}^{3} \varepsilon_{ijk} \frac{\partial}{\partial x^j} \varepsilon_{klm} \frac{\partial A_m}{\partial x^l} + \varepsilon_0 \mu_0 \left[\frac{\partial^2}{\partial t \partial x^i} \phi + \frac{\partial^2}{\partial t^2} A_i \right] = \mu_0 J_i$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \vec{A} - \vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} \right) - \frac{1}{c^2} \frac{\partial}{\partial t} \vec{\nabla} \phi = -\mu_0 \vec{J}$$

Lorenz Gauge

Lorenz Gauge condition is

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$$

Applying this condition yields the following, very symmetrical version of the potential equations

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\phi = -\frac{\rho}{\varepsilon_0}$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{A} = -\mu_0 \vec{J}$$

Energy/Power considerations

Units of electric field , electric displacement, and current density

$$\left[\vec{E}\right] = \frac{\mathrm{Nt}}{C} \qquad \left[\vec{J}\right] = \frac{C}{\sec\cdot \mathrm{m}^2} \qquad \left[\vec{D}\right] = \frac{C}{\mathrm{m}^2}$$

$$\left[\vec{E}\right]\left[\vec{J}\right] = \frac{J}{\sec\cdot \mathrm{m}^3} \qquad \left[\vec{E}\right]\left[\vec{D}\right] = \frac{J}{\mathrm{m}^3}$$

Power delivered from electromagnetic field to a current element per unit volume involves *E* and *J*. Power delivered to a current in a small volume *dV* must be

$$= \left(\vec{E}\cdot\vec{J}\right) dV$$

Use Maxwell Equations to derive general result

Energy Conservation

Energy/volume involves multiplication of *E* and *D*. The exact formula comes from the Maxwell Equations:

$$\int_V \vec{E}\cdot\vec{J}d^3x = \int_V \vec{E}\cdot\left[\nabla\times\vec{H} - \partial\vec{D}/\partial t\right]d^3x$$

$$-\int_V \vec{H}\cdot\left[\nabla\times\vec{E} - \partial\vec{B}/\partial t\right]d^3x$$

$$= -\int_V \nabla\cdot\left[\vec{E}\times\vec{H}\right]d^3x - \frac{\partial}{\partial t}\int_V \frac{\left[\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}\right]}{2}d^3x$$

The RHS terms define the [1] Poynting (Energy Flux) Vector (J/sec m^2)

$$\vec{S} = \vec{E}\times\vec{H}$$

Field Energy Evolution Equation

For linear materials have [2] energy density (J/m^3)

$$u = \left(\frac{\vec{E} \cdot \vec{D}}{2} + \frac{\vec{H} \cdot \vec{B}}{2} \right) = \left(\frac{\varepsilon \vec{E} \cdot \vec{E}}{2} + \frac{\mu \vec{H} \cdot \vec{H}}{2} \right)$$

Field Energy Evolution

$$\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = -\vec{J} \cdot \vec{E}$$

Momentum Conservation

Momentum is a vector quantity. Mechanical momentum delivered to a load by an electromagnetic field is

$$\frac{d\vec{p}_{mech}}{dt} = \int_V \left[\rho \vec{E} + \vec{J} \times \vec{B} \right] dxdydx$$

Define the vector force density to be

$$\left[\rho \vec{E} + \vec{J} \times \vec{B} \right]$$

Momentum Flux density must have nine components

$$\hat{e}_i T_{ij} \varepsilon_{jkl} dx^k \otimes dx^l = \hat{e}_i \left[T_{i1} dy \wedge dz + T_{i2} dz \wedge dx + T_{i3} dx \wedge dy \right]$$

In free space

$$\rho = \varepsilon_0 \vec{\nabla} \cdot \vec{E} \qquad \vec{J} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

then

$$\rho \vec{E} + \vec{J} \times \vec{B} = \varepsilon_0 \begin{bmatrix} \vec{E}\left(\vec{\nabla} \cdot \vec{E}\right) + c^2 \vec{B}\left(\vec{\nabla} \cdot \vec{B}\right) \\ -\vec{E} \times \left(\vec{\nabla} \times \vec{E}\right) - c^2 \vec{B} \times \left(\vec{\nabla} \times \vec{B}\right) \end{bmatrix} - \varepsilon_0 \frac{\partial}{\partial t}\left(\vec{E} \times \vec{B}\right)$$

Momentum change equation

$$\frac{d\vec{p}_{mech}}{dt} + \frac{d\vec{p}_{field}}{dt} =$$

$$\varepsilon_0 \int_V \left[\vec{E}\left(\vec{\nabla} \cdot \vec{E}\right) + c^2 \vec{B}\left(\vec{\nabla} \cdot \vec{B}\right) - \vec{E} \times \left(\vec{\nabla} \times \vec{E}\right) - c^2 \vec{B} \times \left(\vec{\nabla} \times \vec{B}\right) \right] dxdydz$$

Maxwell Stress Tensor

Momentum in field

$$\vec{p}_{field} = \frac{1}{c^2}\int_V \vec{E} \times \vec{H} dxdydz$$

Stress Tensor

$$T_{ij} = \varepsilon_0 \left[E_i E_j + c^2 B_i B_j - \frac{1}{2}\left(\vec{E}\cdot\vec{E} + c^2 \vec{B}\cdot\vec{B}\right)\delta_{ij} \right] =$$

$$\varepsilon_0 \begin{pmatrix} E_x E_x + c^2 B_x B_x & E_x E_y + c^2 B_x B_y & E_x E_z + c^2 B_x B_z \\ E_y E_x + c^2 B_y B_x & E_y E_y + c^2 B_y B_y & E_y E_z + c^2 B_y B_z \\ E_z E_x + c^2 B_z B_x & E_z E_y + c^2 B_z B_y & E_z E_z + c^2 B_z B_z \end{pmatrix} - \frac{\varepsilon_0}{2}\begin{pmatrix} \vec{E}\cdot\vec{E} + \\ c^2 \vec{B}\cdot\vec{B} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$