TAAD1
HW Problem Set 1

1. For the harmonic oscillator potential

$$V(x) = m\omega^2 x^2 / 2.$$

The Lagrangian is

$$L = T - V = m\frac{\dot{x}^2}{2} - m\omega^2 \frac{x^2}{2}.$$

The canonical momentum is

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}. \qquad \therefore \dot{x} = \frac{p}{m}$$

Therefore, the Hamiltonian is

$$H = \dot{x}p - L\left(\dot{x}(x, p), x\right) = \frac{p^2}{m} - \frac{p^2}{2m} + m\omega^2 \frac{x^2}{2}$$

$$= \frac{p^2}{2m} + m\omega^2 \frac{x^2}{2}$$

Hamilton's equations are

$$\dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x \qquad \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

Taking the time derivative of the second equation gives

$$\ddot{x} = \frac{\dot{p}}{m} = -\omega^2 x,$$

the harmonic oscillator equation. Because this equation can be written in a Hamiltonian form, Liouville's theorem applies. The motion preserves phase space volume (area in the 1-D case).

2. The lagrangian for a particle in an E-M field is

$$L\left(\dot{\vec{x}}, \vec{x}\right) = m\frac{\dot{\vec{x}} \cdot \dot{\vec{x}}}{2} - q\phi(\vec{x}, t) + q\dot{\vec{x}} \cdot \vec{A}(\vec{x}, t)$$

The canonical momentum, $\vec{P}$, is

$$\vec{P} = \frac{\partial L}{\partial \dot{\vec{x}}} = m\dot{\vec{x}} + q\vec{A}(\vec{x}, t)$$

The Euler-Lagrange equation is

$$\frac{d\vec{P}}{dt} = \frac{d}{dt}\left[m\dot{\vec{x}} + q\vec{A}(\vec{x}, t)\right] = \frac{\partial L}{\partial \vec{x}} = \nabla\left[-q\phi(\vec{x}, t) + q\dot{\vec{x}} \cdot \vec{A}(\vec{x}, t)\right]$$

$$m\ddot{\vec{x}} = \vec{F} = q\left[-\nabla\phi - \frac{\partial \vec{A}}{\partial t} - \dot{\vec{x}} \cdot \frac{\partial \vec{A}}{\partial \vec{x}} + \nabla\left(\dot{\vec{x}} \cdot \vec{A}(\vec{x}, t)\right)\right].$$

To proceed, use one of two methods:

[1] There is a vector identity

$$\nabla\left(\vec{a}\cdot\vec{b}\right)=\left(\vec{a}\cdot\nabla\right)\vec{b}+\left(\vec{b}\cdot\nabla\right)\vec{a}+\vec{a}\times\left(\nabla\times\vec{b}\right)+\vec{b}\times\left(\nabla\times\vec{a}\right)$$
$$=\left(\vec{a}\cdot\nabla\right)\vec{b}+\vec{a}\times\left(\nabla\times\vec{b}\right),$$

when $\vec{a}$ is a constant during the $\vec{x}$ differentiation. Then the final terms in the force equation reduce to

$$\vec{F}=q\left[-\nabla\phi-\frac{\partial\vec{A}}{\partial t}+\dot{\vec{x}}\times\left(\nabla\times\vec{A}\left(\vec{x},t\right)\right)\right]$$
$$=q\left[\vec{E}+\vec{\mathrm{v}}\times\vec{B}\right]$$

[2] Expand the final terms in the force equation by hand. For example, the x-component of these terns is

$$\frac{\partial}{\partial x}\left(\mathrm{v}_x A_x+\mathrm{v}_y A_y+\mathrm{v}_z A_z\right)-\left(\mathrm{v}_x\frac{\partial}{\partial x}+\mathrm{v}_y\frac{\partial}{\partial y}+\mathrm{v}_z\frac{\partial}{\partial z}\right)A_x$$
$$=\mathrm{v}_y\frac{\partial A_y}{\partial x}+\mathrm{v}_z\frac{\partial A_z}{\partial x}-\mathrm{v}_y\frac{\partial A_x}{\partial y}-\mathrm{v}_z\frac{\partial A_x}{\partial z}=\mathrm{v}_y B_z-\mathrm{v}_z B_y$$
$$=\left(\vec{\mathrm{v}}\times\vec{B}\right)_x$$

3. (a) The loop integrals are

$$\oint ydx=\int_0^1 bdx+0+\int_1^0 adx+0$$
$$=b-a$$

(b)

$$\oint ydx=\int_0^1(1-x)dx+\int_1^0 0dx+0$$
$$=\left.\left(x-\frac{x^2}{2}\right)\right|_0^1=\frac{1}{2}$$

(c)

$$\oint ydx=\int_{-1}^{1}\sqrt{1-x^2}\,dx+\int_{1}^{-1}-\sqrt{1-x^2}\,dx$$
$$=\int_\pi^0\sin\theta(-\sin\theta)d\theta+\int_0^\pi-\sin\theta(-\sin\theta)d\theta$$
$$=2\frac{\pi}{2}=\pi$$

(d) It is easy to reparameterize (c) in terms of $\theta\in[0,2\pi]$ directly

$$x=\cos(-\theta)=\cos\theta\qquad y=\sin(-\theta)=-\sin\theta$$
$$\oint ydx=\int_0^{2\pi}-\sin\theta\left(d\cos\theta\right)=\int_0^{2\pi}\sin^2\theta d\theta$$

$$= 2\pi \frac{1}{2} = \pi$$

$-\theta$ is used in the functional forms to get the clockwise sense of circulation.