Final Examination
Accelerator Physics
Due May 9, 2011

1. CERN's Large Hadron Collider (LHC) accelerates protons to very high energy. In the design document the design energy is 7 TeV, or 7000 GeV.
    a. What is the relativistic γ of the accelerated protons? What is their relativistic β?
    b. What is their magnetic rigidity?
    c. There are 1232 dipole magnets of length 14.3 m, bending in normal configuration. What is the required dipole magnetic field?
    d. What is the bend radius when the particles are in the dipoles? Is your result consistent with the total reported machine circumference of 26658.883 m? Explain.
    e. What is the revolution frequency?
    f. The RF frequency is reported to be 400.8 MHz. What is the harmonic number to four significant digits?
    g. The total energy gain per pass is 8 MeV. How long (seconds) does it take to accelerate from 450 GeV to 7 TeV?

    You can find the answers to many of these questions in the design document https://edms.cern.ch/file/445830/5/Vol_1_Chapter_2.pdf
    Of course, you will be graded on your approach to calculating the answers!

2. In colliders, to increase the event rate, one would like the beam size very small at the collision point. This leads to the notion of a "low beta insertion". A (grossly!) simplified model of such an insertion is a strong thin focusing lens of focal length f followed by a drift. Suppose $s=0$ at the location of the lens, $\beta(s=0)=\beta_0$ is given, and the ellipse is upright (what does this imply about $\alpha(s=0)$?).
    a. What is $\gamma(s=0)$ for the ellipse?
    b. What is the transfer matrix to a location of distance s downstream of the focusing lens?
    c. What is the inverse of the matrix?
    d. What is $\beta(s)$ (get this from the ellipse transformation formula)?
    e. Show $\beta(s)$ is an extremum at location
    $$s_{extremum}=\frac{\beta_0^2}{f\left(1+\beta_0^2/f^2\right)}.$$
    f. Is it a maximum or minimum?
    g. What is $\beta^*\equiv\beta\left(s_{extremum}\right)$, the value at the location of the extremum?
    h. Show the result from part d can be written in the form
    $$\beta(s)=\beta^*+\frac{\left(s-s_{extremum}\right)^2}{\beta^*}$$

Hint: Complete the square.

i. What is $\alpha\left(s_{extremum}\right)$, calculated by the ellipse transformation formula?

j. Extra Credit: Why does $\alpha\left(s_{extremum}\right)$ have this value?

3. Suppose a one period 2 by 2 transfer matrix has $\left|\mathrm{Tr}\left(M\right)\right| > 2$.
   a. Find an expression for the two eigenvalues in terms of $\mathrm{Tr}\left(M\right)$. Call them λ_+ and λ_-.
   b. Are the two eigenvalues real, or do they have some imaginary components?
   c. Show $\lambda_+\lambda_- = 1$.
   d. By linear algebra, the matrix of the eigenvectors, S, diagonalizes the matrix M
   $$\begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} = SMS^{-1}$$
   Give an expression for M^n in terms of this diagonal matrix and S.
   e. Is M bounded as n increases?
   f. Comment on the stability or lack of stability of solutions given by repeated application of the transfer matrix M.
   g. Using $\det\left(AB\right) = \det\left(A\right)\det\left(B\right)$, the formula for the determinant of the inverse of a matrix, and part d, comment on the property of the transport matrix that ensures part c is true.
   h. Show in general (most straightforwardly by completing the matrix multiplication) that
   $$\mathrm{Tr}\left(SMS^{-1}\right) = \mathrm{Tr}\left(M\right)$$
   as long as S is an invertible matrix.

4. Note the following expressions for the transfer matrix of a thin lens alternating gradient FODO system. The one-period transfer matrix starting with the middle of the focusing magnet is
$$M_f = \begin{pmatrix} 1 & 0 \\ -1/\left(2f\right) & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/\left(2f\right) & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 - L^2/\left(2f^2\right) & 2L + L^2/f \\ -L/\left(2f^2\right) + L^2/\left(4f^3\right) & 1 - L^2/\left(2f^2\right) \end{pmatrix}.$$
and M_d is obtained from M_f by replacing f with $-f$
   a. How should one choose f, in terms of L, so that the phase advance through one period of the FODO system is 60 degrees (corresponding to 1/6 of a transverse oscillation per period)?

b. For the matched phase space (x, x') ellipse in the 60 degree phase advance system, what is the beta-function in the middle of the focusing lens?
c. For the matched phase space ellipse in the 60 degree phase advance system, what is the beta-function in the middle of the defocusing lens?
d. What are the alpha-functions for the matched ellipses at these same two locations?
e. Suppose L is 2 m, and the area of a matched phase space ellipse is $\pi\varepsilon = 10\pi \times 10^{-6}$ m radian, what are the maximum extents ($x_{\max}$) of the matched ellipses in the focusing and defocusing lenses of the 60 degrees phase advance system?

5. In this problem you are asked to show that for constant uniform focusing, with solutions to Hill's equation given by constant frequency sinusoids, the notion of matching is simplified.
   a. Show the transfer matrix from s to s' for a region with uniform focusing strength $k(s) = k$ is

$$M(s', s) = \begin{pmatrix} \cos\left(\sqrt{k}\,(s'-s)\right) & \sin\left(\sqrt{k}\,(s'-s)\right)/\sqrt{k} \\ -\sqrt{k}\sin\left(\sqrt{k}\,(s'-s)\right) & \cos\left(\sqrt{k}\,(s'-s)\right) \end{pmatrix}.$$

   b. What is the phase advance of this matrix?
   c. What are the ellipse parameters $\alpha(s), \beta(s), \gamma(s)$ of the matched phase-space ellipses?
   d. Given a matched phase ellipse entering the region, what are $\alpha(s'), \beta(s'), \gamma(s')$? (Hint: Use the transformation formula.)
   e. Interpret.
   f. Suppose $\beta(s) = \beta_{matched}(s) + \Delta\beta$ is slightly above the matched value found in part c, what is $\beta(s')$?
   g. Interpret.