Physics 319
Classical Mechanics

G. A. Krafft
Old Dominion University
Jefferson Lab

Lecture 8

Path Independence of Coulomb Force

- Coulomb force is curl-free (conservative)

$$\vec{F} = k\frac{x\hat{x} + y\hat{y} + z\hat{z}}{r^3}$$

$$\text{curl } \vec{F} = k\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{x}{r^3} & \dfrac{y}{r^3} & \dfrac{z}{r^3} \end{vmatrix} = \begin{matrix} k\hat{x}\left(-\dfrac{3zy}{r^5} + \dfrac{3yz}{r^5}\right) \\ +k\hat{y}\left(-\dfrac{3xz}{r^5} + \dfrac{3zx}{r^5}\right) \\ +k\hat{z}\left(-\dfrac{3yx}{r^5} + \dfrac{3xy}{r^5}\right) \end{matrix} = 0$$

$$U(\vec{r}) = \frac{k}{r}$$

Time Dependent Potential

- If force curl free at every time *t*

$$U\left(\vec{r},t\right) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}\left(\vec{r}',t\right)d\vec{r}'$$

$$\frac{dU}{dt} = -\vec{F}\cdot\vec{v} + \frac{\partial U}{\partial t}$$

$$\frac{dT}{dt} = \vec{F}\cdot\vec{v}$$

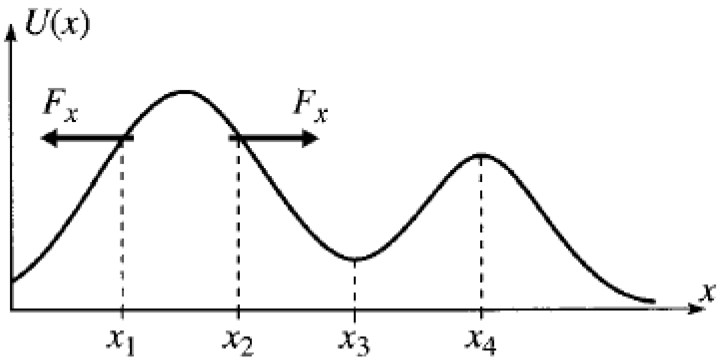
- Modified Energy Conservation Rule

$$\frac{dE}{dt} = \frac{\partial U}{\partial t}$$

One Dimensional Potential

- Potential and Force

$$U(x) = -\int_{x_0}^{x} F_x(x')\,dx' \qquad F_x(x) = -\frac{dU}{dx}$$



- Always solvable (perhaps numerically)

$$t = \int_{x_0}^{x} \frac{dx'}{\dot{x}'} = \sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{dx'}{\sqrt{E - U(x')}}$$

Spherical Coordinates

- Gradient in

$$\vec{\nabla} f \equiv \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

- For spherically symmetric forces

$$\vec{F} = -\hat{r}\frac{\partial U}{\partial r}$$

- Conservative iff spherically symmetric and then

$$U(r) = -\int_{r_0}^{r} F_r(r')\,dr'$$

Interaction Potential

- Consider two gravitationally interacting particles

$$F_{12} = -\frac{Gm_1 m_2}{\left|\vec{r}_1 - \vec{r}_2\right|^3}\left(\vec{r}_1 - \vec{r}_2\right) = -F_{21}$$

$$\therefore F_{ab} = -\frac{Gm_1 m_2}{\left|\vec{r}_a - \vec{r}_b\right|^3}\left(\vec{r}_a - \vec{r}_b\right)$$

- Symmetric form translationally invariant and force form we know is conservative for each particle individually. So the force on *both*(!) particles can be written as the gradient of the *common* interaction potential

$$F_{ab} = -\nabla_a U\left(\vec{r}_a - \vec{r}_b\right) = \nabla_a \frac{Gm_1 m_2}{\left|\vec{r}_a - \vec{r}_b\right|} \qquad U\left(\vec{r}_a - \vec{r}_b\right) = -\frac{Gm_1 m_2}{\left|\vec{r}_a - \vec{r}_b\right|}$$

Energy Conservation

- The conserved energy involves the interaction potential

$$E = T_1 + T_2 + U\left(\vec{r}_1 - \vec{r}_2\right)$$

$$\frac{dE}{dt} = \vec{F}_{12} \cdot \vec{v}_1 + \vec{F}_{21} \cdot \vec{v}_2 + \frac{\partial U}{\partial \vec{r}} \cdot \frac{d\vec{r}_1}{dt} - \frac{\partial U}{\partial \vec{r}} \cdot \frac{d\vec{r}_2}{dt}$$

$$= \vec{F}_{12} \cdot \vec{v}_1 + \vec{F}_{21} \cdot \vec{v}_2 - \vec{F}_{12} \cdot \vec{v}_1 + \vec{F}_{12} \cdot \vec{v}_2$$

$$= \vec{F}_{12} \cdot \vec{v}_1 + \vec{F}_{21} \cdot \vec{v}_2 - \vec{F}_{12} \cdot \vec{v}_1 - \vec{F}_{21} \cdot \vec{v}_2$$

$$= 0$$

Generalization to Many Bodies

- If both the interaction forces and the external forces are conservative with interaction potential and external potential given

$$T = \sum_{\alpha} T_{\alpha}$$

$$U = \sum_{\alpha} \sum_{\beta > \alpha} U_{int}\left(\vec{r}_{\alpha} - \vec{r}_{\beta}\right) + \sum_{\alpha} U_{ext}\left(\vec{r}_{\alpha}\right)$$

$$\frac{dE}{dt} = \frac{dT}{dt} + \frac{dU}{dt} = 0$$

- In a *Rigid Body*, distances between body particles are fixed and changes in the interaction energy are negligibly small

Elastic Collisions

- Two particles of mass m collide elastically (no energy loss)

$$T = \frac{m}{2} v_1^2 = \frac{m}{2} v_1'^2 + \frac{m}{2} v_2'^2$$

$$m\vec{v}_1 = m\vec{v}_1' + m\vec{v}_2' \rightarrow \vec{v}_1 = \vec{v}_1' + \vec{v}_2'$$

$$\frac{m}{2}\left(v_1'^2 + 2\vec{v}_1' \cdot \vec{v}_2' + v_2'^2\right) = \frac{m}{2} v_1'^2 + \frac{m}{2} v_2'^2$$

$$\therefore \vec{v}_1' \cdot \vec{v}_2' = 0$$