NICA: Conceptual Proposal for Collider

Valeri Lebedev
Fermilab

Jefferson Lab
January 27, 2011
(First presented at MAC in Dubna, January 11, 2010)

Disclaimer

- This report was presented at MAC in JINR (Dubna, Russia) one year ago to show interdependency of different machine parameters
- It is not necessarily reflects the present status of NICA design
- Minor corrections were introduced to the viewgraphs to reflect improvements of our understanding

Objectives for the collider ring proposal

- Maximize the luminosity basing on experience obtained during the Tevatron Run II commissioning and upgrades
- Look into optimal strategy
  - Parameter interdependence
  - Major effects limiting the luminosity
    - Luminosity lifetime
    - IBS
    - Optics
    - Cooling (stochastic cooling is sufficient)
    - Instabilities
- Only operation with heavy ions are considered (Au, Z=79, A=197)
  - Optics is optimized for 4.5 GeV/n
  - Proton mode requires additional insight and may be additional place in straight lines
    - Polarization
    - Electron cooling

Main Collider Parameters

	1 GeV/n	3.5 GeV/n	4.5 GeV/n
Ring circumference, m	454		
Momentum compaction	0.02585		
Betatron tunes	9.46		
Number of bunches		20	
Particle per bunch	$0.28 \cdot 10^9$	$3.89 \cdot 10^9$	$5.3 \cdot 10^9$
Beam current, A	0.041	0.63	0.872
Ring acceptance, mm mrad	40		
Ring long. acceptance, $\Delta p/p$	± 0.0125		
RMS emittance, $\varepsilon_x/\varepsilon_y$	1.1/0.855	1.12/0.69	1.12/0.595
RMS momentum spread	$0.695 \cdot 10^{-3}$	$1.56 \cdot 10^{-3}$	$1.79 \cdot 10^{-3}$
Maximum tune shift ($\delta\nu_{SP} + 2\delta\nu_{BB}$)	0.05		
IP beta-function, cm	35		
Bunch length, cm	63	72 (5σ)	60 (6σ)
RF voltage, kV	760	760	370
IBS growth time, s	300	1000	1100
Upper boundary of stochastic cooling band, GHz	0.25	1.5	3
Cooling time, s	72	130	230
Luminosity, $cm^{-2}\ s^{-1}$	$1.2 \cdot 10^{25}$	$2.7 \cdot 10^{27}$	$6 \cdot 10^{27}$

- The same acceptance and maximum tune shifts are assumed for all energies
- Optics is optimized for 4.5 GeV/n
- Circumference was increased to ~540 m (as January of 2011)

Luminosity Limitation due to Beam Space Charge

- For bunched beam with Gaussian distribution in all planes

$$\begin{bmatrix}\delta\nu_{SC_X}\\ \delta\nu_{SCY}\end{bmatrix}=\frac{r_p Z^2 N_i}{2\pi A\beta^2\gamma^3}\frac{C}{\sqrt{2\pi}\sigma_s}\left\langle\frac{1}{(\sigma_x+\sigma_y)}\begin{bmatrix}\beta_x/\sigma_x\\ \beta_y/\sigma_y\end{bmatrix}\right\rangle_s, \quad \sigma_{x,y}=\sqrt{\beta_{x,y}\varepsilon_{x,y}+\left(D_{x,y}\sigma_p\right)^2}$$

  - For round beam, smooth focusing and sufficiently small dispersion

=> $$\delta\nu_{SC}=\frac{r_p Z^2 N_i}{4\pi A\beta^2\gamma^3\varepsilon}\frac{C}{\sqrt{2\pi}\sigma_s}$$

    - Weak dependence of the tune shifts on optics
    - It limits the beam longitudinal density, N_i/σ_s

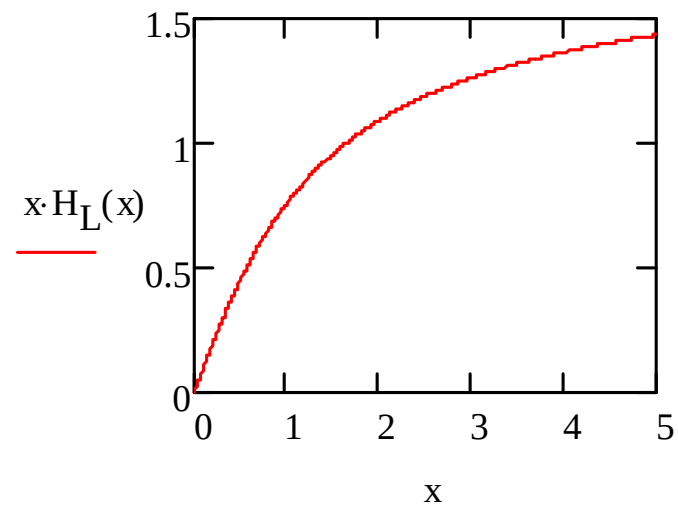
- For $\beta_x^* = \beta_y^*$ and head-on collisions the luminosity is

$$L=\frac{f_0 n_b N_i^2}{4\pi\beta^*\sqrt{\varepsilon_x\varepsilon_y}}H_L\left(\frac{\sigma_s}{\beta^*}\right), \qquad H_L(x)=\frac{2}{\sqrt{\pi}}\int_0^\infty\frac{e^{-y^2}dy}{1+x^2y^2}$$

- Combining the above equations and assuming $\varepsilon_x = \varepsilon_y$ one obtains a luminosity limitation

$$\boxed{L=\frac{\sqrt{2\pi}A\beta^2\gamma^3}{r_p Z^2}\frac{f_0 N_i}{(C/n_b)}\left(\frac{\sigma_s}{\beta^*}H_L\left(\frac{\sigma_s}{\beta^*}\right)\right)\delta\nu_{SC}}$$

Longer bunch makes more luminosity!!!



Luminosity Limitation due to Beam Space Charge (2)

- For the case of fixed ring acceptance and circumference one should also exclude N_i. That results in

$$L = 8\pi^2 \frac{A^2 c \beta^5 \gamma^6}{Z^4 r_p^{\ 2}} \frac{n_b \sigma_s \varepsilon}{C^3} \left(\frac{\sigma_s}{\beta^*} H\left(\frac{\sigma_s}{\beta^*} \right) \right) \delta\nu_{SC}^{\ 2} \xrightarrow{C \propto \beta\gamma} \propto \beta^2 \gamma^3$$

- Increase of maximum machine energy increases circumference and, consequently, reduces the luminosity for the lower energies
- The beam space charge creates the major luminosity limitation
- Large luminosity requires
  - Small circumference, C
  - Short separation length, C/n_b
  - Large value of σ_s/β^*
  - Large emittance => large acceptance

  - Means to keep the beam stable with large beam current

Luminosity Limitation due to Beam-beam Effects

- For bunched beam with Gaussian distribution

$$\begin{bmatrix} \delta\nu_{BB_X} \\ \delta\nu_{BBY} \end{bmatrix} = \frac{r_p Z^2 N_i}{4\pi A\beta^2\gamma} \frac{1+\beta^2}{\sqrt{\varepsilon_x}+\sqrt{\varepsilon_y}} \begin{bmatrix} 1/\varepsilon_x \\ 1/\varepsilon_y \end{bmatrix}$$

  - For round beam => $$\delta\nu_{BB} = \frac{r_p Z^2 N_i}{4\pi A\beta^2\gamma\varepsilon}\left(\frac{1+\beta^2}{2}\right)$$

- Combining the above equation with the equation for space charge tune shift one obtains

$$\boxed{\delta\nu_{BB} = \delta\nu_{SC}\sqrt{\frac{\pi}{2}}\frac{\sigma_s}{C}\gamma^2\left(1+\beta^2\right)}$$

- For NICA parameters the space charge tune shift is significantly smaller than the tune shift due to beam space charge
- Small β^* results small σ_s and, consequently, small $\delta\nu_{BB}$
  - We need small β^* only because it reduces $\delta\nu_{BB}$ (no direct effect of β^* on the luminosity)
- Large value of σ_s/β^* results in phase averaging for high order resonances and significantly mitigates the beam-beam effects

Luminosity Lifetime

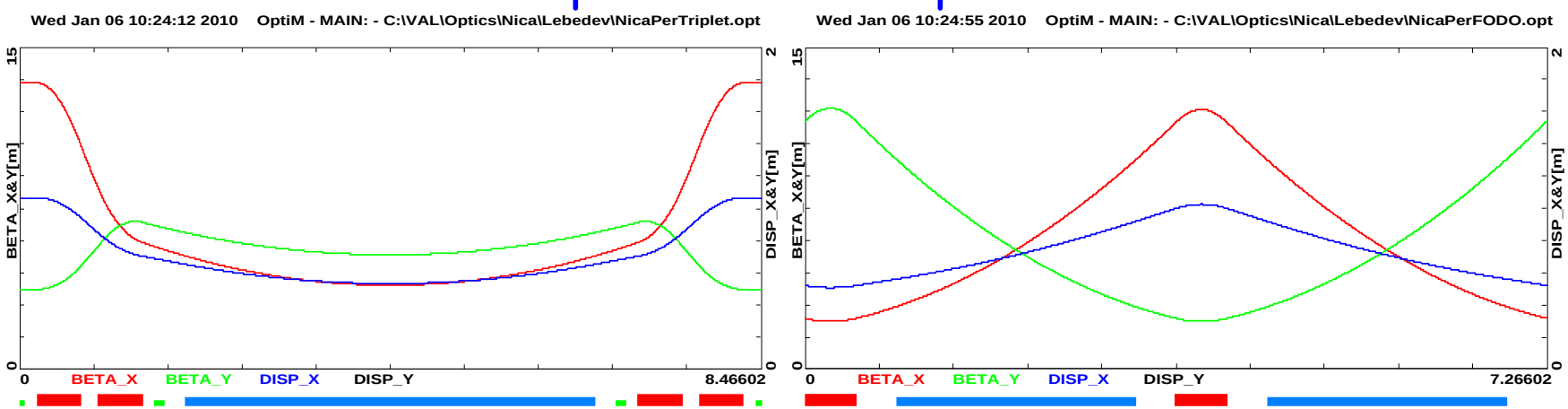
- Cross-sections of particle loss for Au-Au collisions

Nuclear, Au-Au,	7 barn
Electro-magnetic, $\theta > 12$ mrad (6σ)	70 mbarn

  - ⇨ Intensity lifetime – 360 hour for 2 IPs and $L=7.1\cdot10^{27}$ $cm^{-2}s^{-1}$
  - ⇨ Events per collision – $3.6\cdot10^{-3}$ (47 kHz)
  - ⇨ Long stores (>24 h) if other mechanisms of particle loss can be made insignificant
- Sources of particle loss
  - Scattering on the residual gas
  - Capture of residual gas electrons
    - Looks small
  - Electron capture in the electron cooler (~1 hour)
    - We do not really need electron cooling for ions
- Vacuum loss
  - Tevatron beam lifetime ~1000 hour
  - Crossections for Au interaction with gas are ~10 times larger
  - $<2\cdot10^{-9}$ Torr H_2 equivalent averaged over ring should is required (200 h)

Intrabeam Scattering

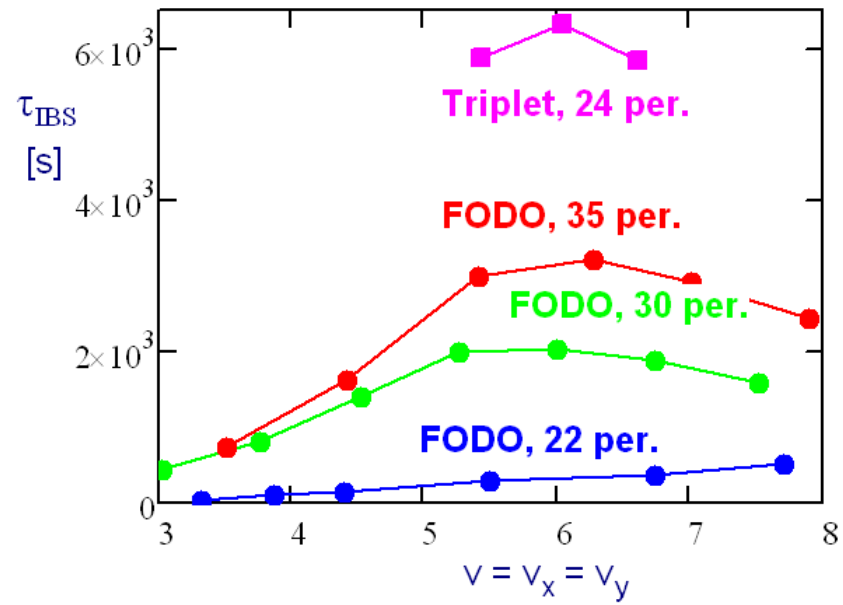
- Smooth focusing
  - System comes to thermal equilibrium below transition
  - Infinite ε growth for H & L degrees of freedom above transition
    - Local L temperature is always smaller H temperature
- Quad focusing
  - Additional emittance growth related to variations of β-functions
- Mitigation of IBS requires
  - Smooth focusing
  - Operation below transition
- Lattices with FODO and triplet focusing were tested
  - Number of cells and phase advance per cell were varied

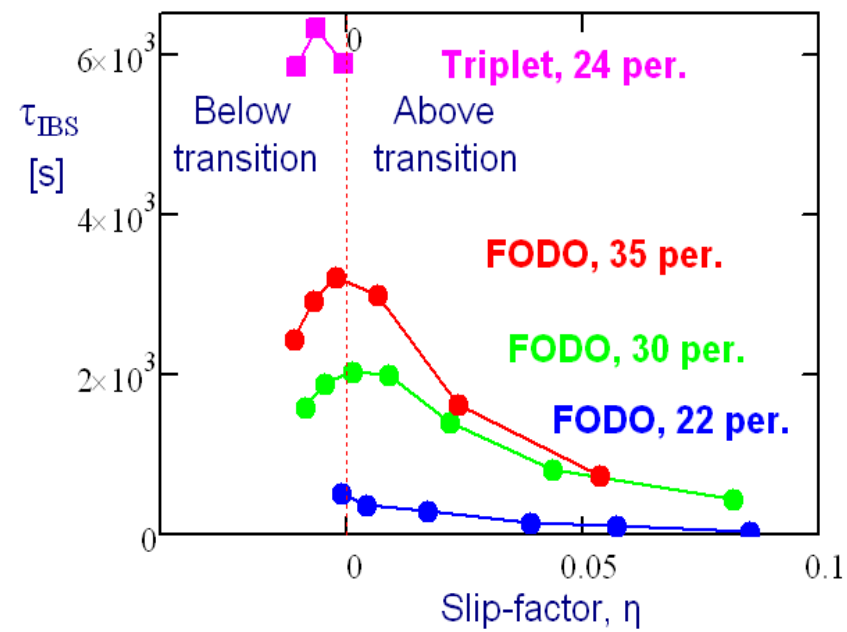


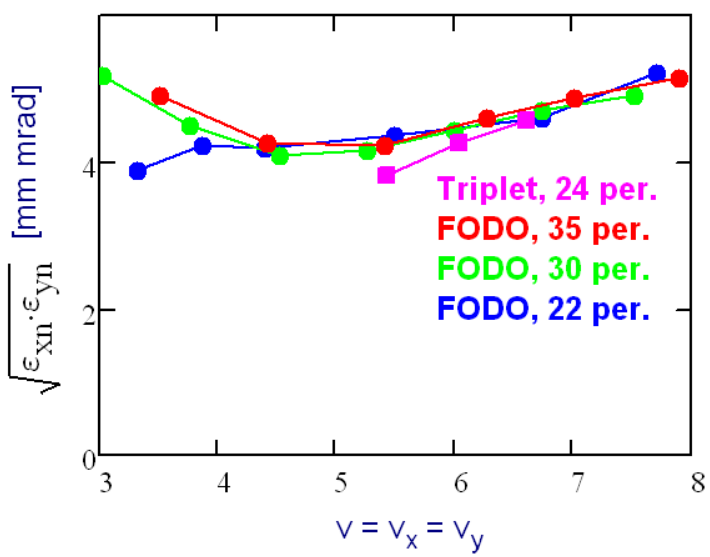
30 periods, 90 deg. per cell

Results of IBS Tests

- ε_x, ε_y and σ_p were adjusted so that to keep equal all 3 growth times and to keep constant value for $\delta\nu_{SC_V}$ (larger one)
  - Resulting $\varepsilon_x\varepsilon_y$ has weak dependence on other parameters
- Minimum heating is at $\gamma \approx \gamma_{cr}$
  - Below transition – i.e. large γ_{cr}
    - large $\Delta\nu$ per cell => strong heating
  - Above transition – i.e. small γ_{cr}
    - Heating due to ΔT between H & L planes







- Triplet focusing looks preferable. It results in doubling IBS growth time

Objectives for Optics Design

$$L = 8\pi^2 \frac{A^2 c \beta^5 \gamma^6}{Z^4 r_p^{\ 2}} \left(\frac{n_b}{C}\right)\left(\frac{\sigma_s}{C}\right)\left(\frac{\varepsilon}{C}\right)\left(\frac{\sigma_s}{\beta^*} H\left(\frac{\sigma_s}{\beta^*}\right)\right) \delta\nu_{SC}^{\ 2}$$

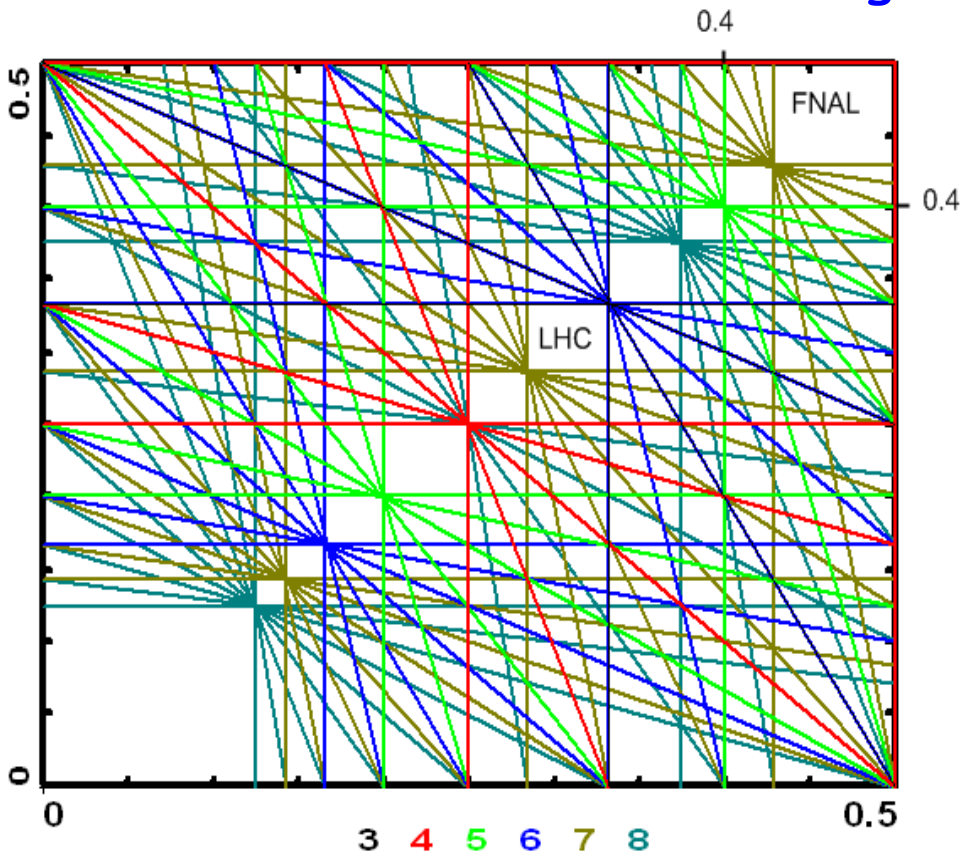
- Small β^*
  - Requires correction of second order chromaticity
- Large acceptance
  - It is limited by chromaticity correction sextupoles
    - It has to be mitigated by appropriate phase advances
- Large momentum acceptance
  - "Thermal equilibrium" suppresses IBS but requires large $\Delta p/p$
  - Reduction of β^* decreases momentum acceptance
- Small circumference
  - Large aperture and large dispersive contribution to horizontal size push to superferic dipoles and quads (2 T field)
    - Cold vac. chamber addresses good vacuum and low wall resistivity
- Collider tune has to be at right place
- Beam separation in IPs
  - Vertical beam separation
  - Accounting of focusing difference in IPs for two proton beams

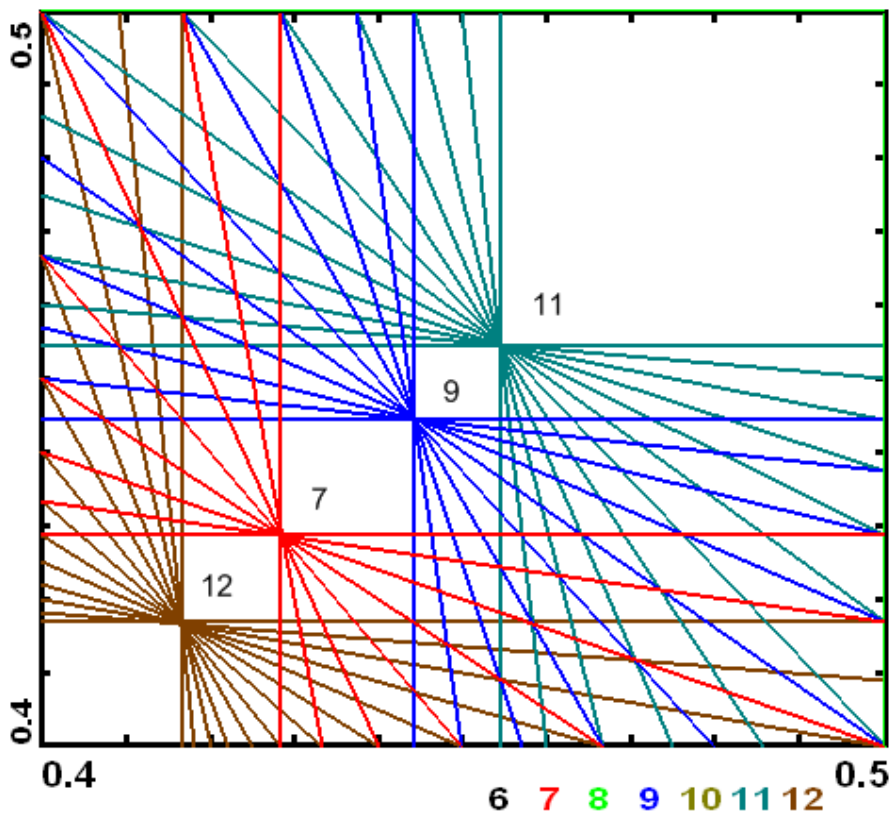
Design Choices for Optics

- Racetrack with 2 IPs
- ±4 m for particle detector
  - Allows to keep sufficiently small β^* and IP contribution to chromaticity
- Triplet focusing through the entire machine (including IPs)
  - Smooth lattice
- Phase advance 90 deg. per cell
  - Cancelation of nonlinearity of sextupoles located at $\delta\nu$=180 deg.
  - Four families of sextupoles (second order chromaticity corr.)
- Dispersion zeroing in straight sections with dipoles of half length
- Vertical beam separation
  - Vertical dispersion suppression with two step orbit elevation
- Tunes ~x.46 (same as Recycler)
  - Inversion of Tevatron tunes (~0.582 -> 0.418)

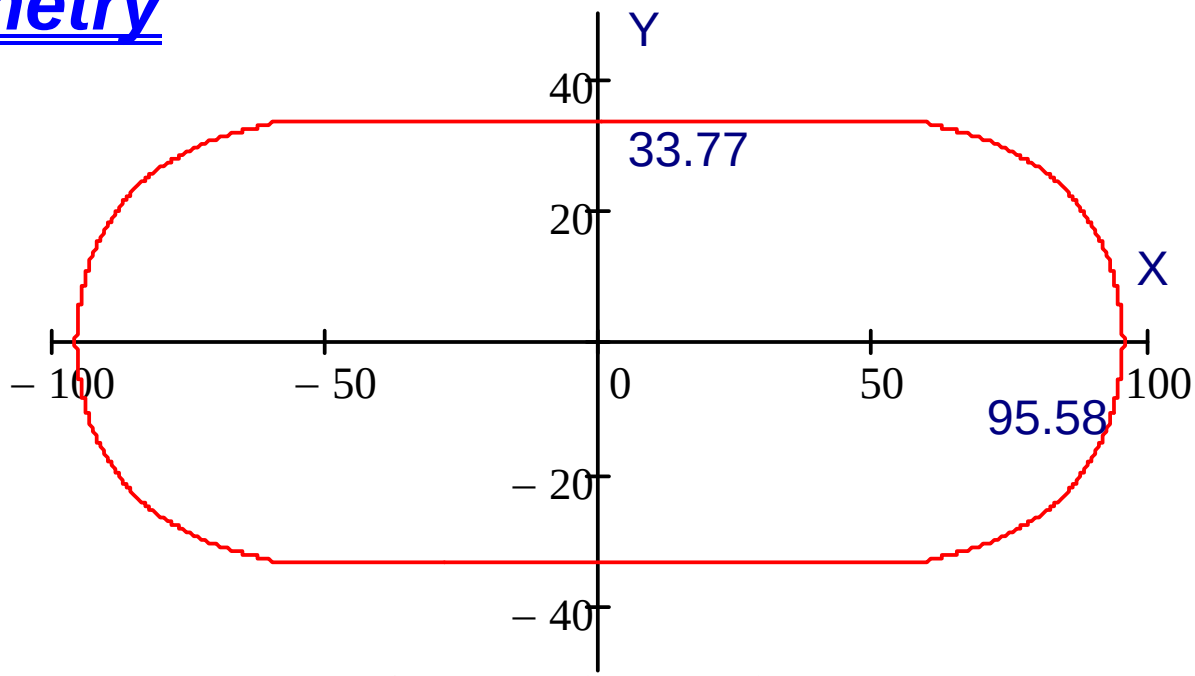
Collider Tune

- Odd resonances are suppressed in the absence of parasitic collisions
  - Tevatron suffers from 7-th order (parasitic collisions) and 12-th order (will be suppressed by cooling in NICA)
- Other possibility is operation near zero
  - Requires two more (or less) 90 deg. periods
  - Bad for stochastic cooling due to overlap ⊥ and L bands

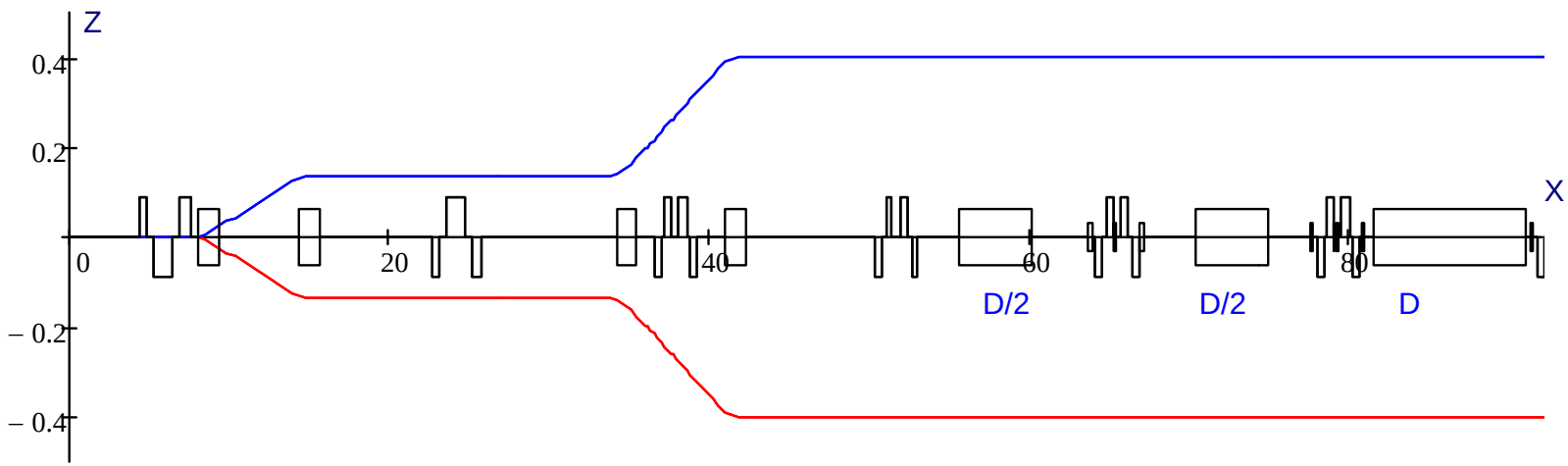




Ring Geometry



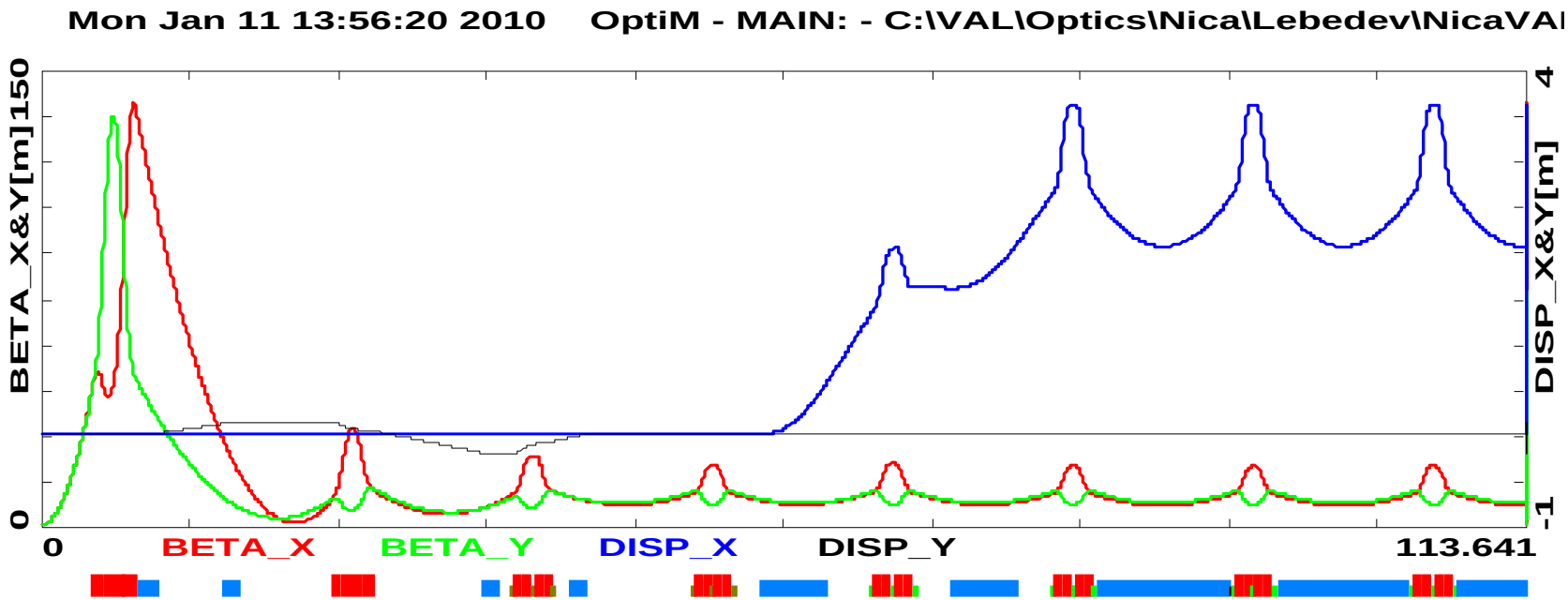
Ring footprint (top view), C = 454 m



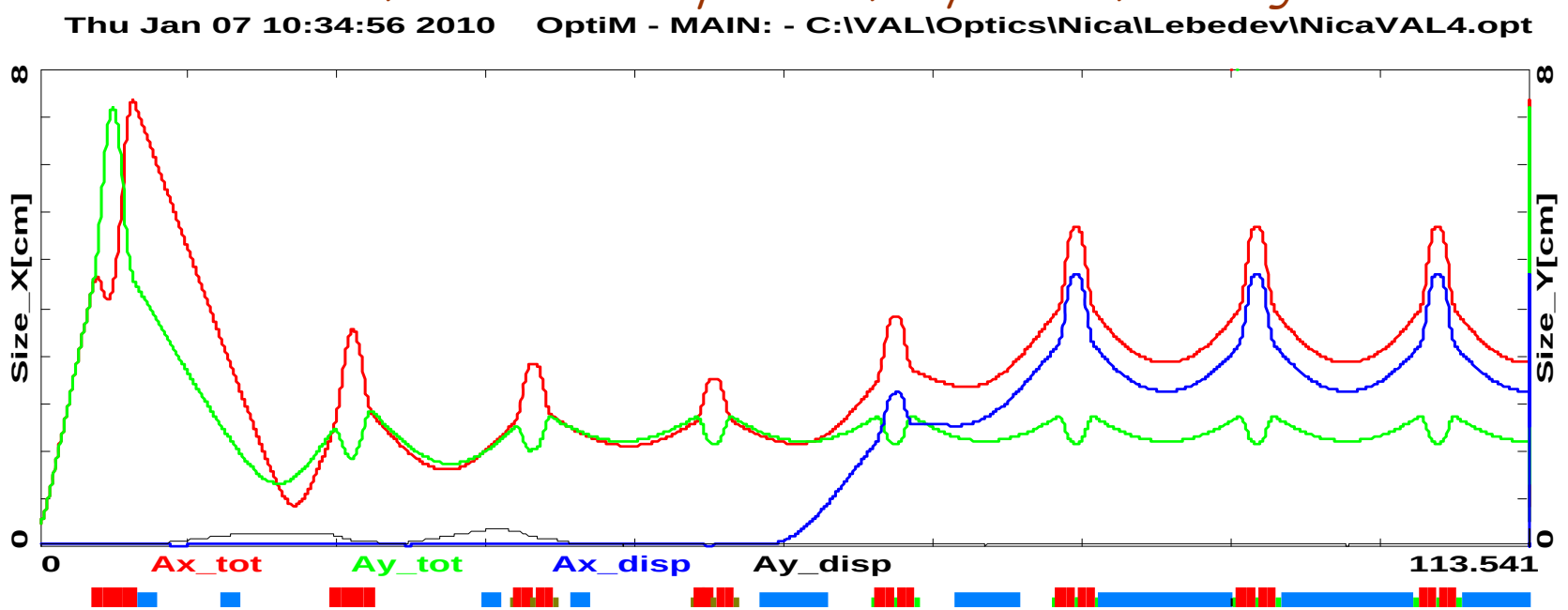
Vertical beam separation in IP

(Larger initial separation (> ±40 cm) would require longer straight sections for suppression of D_v)

Optics

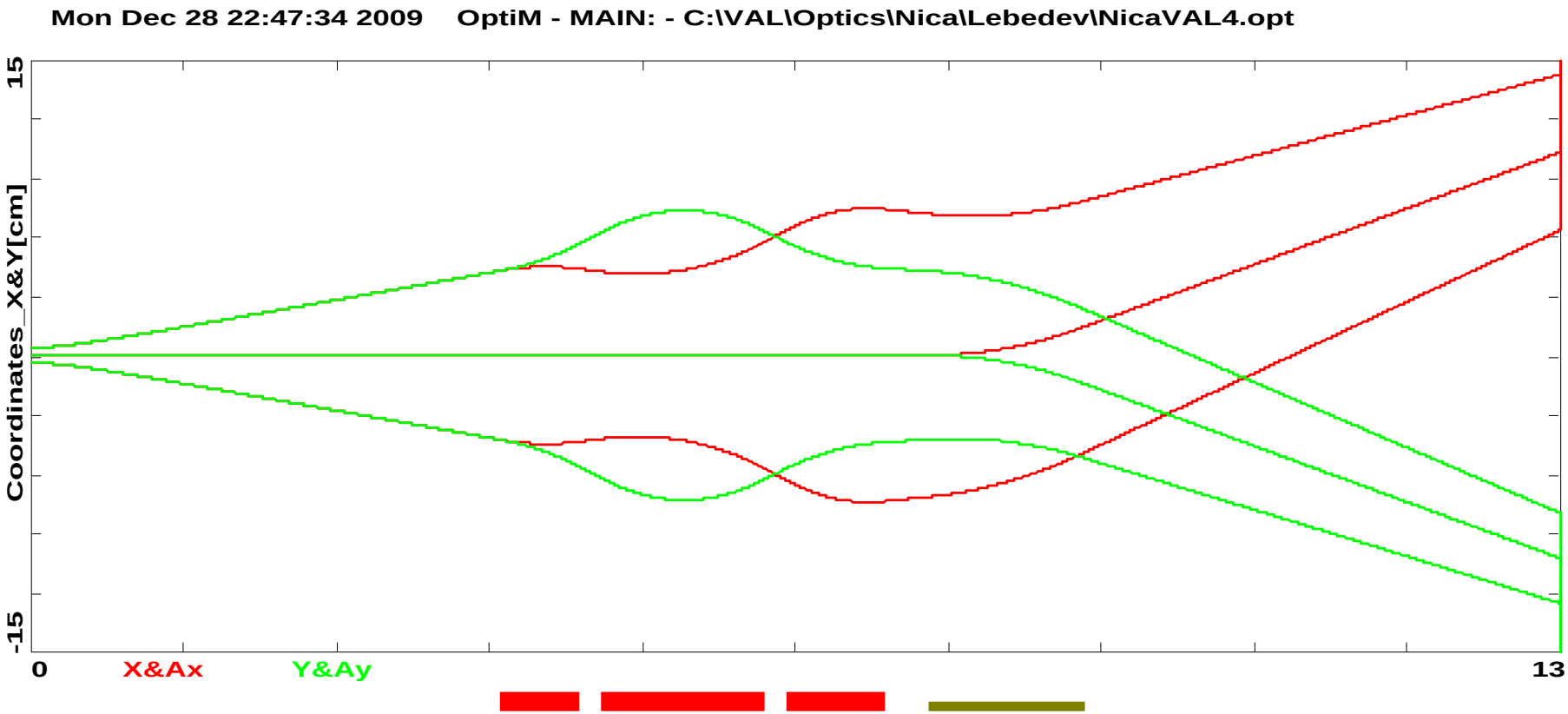


Beta-functions and dispersions for quarter of the ring



Beam envelopes for e=40 mm mrad, Dp/p=1.25%

Beam Separation



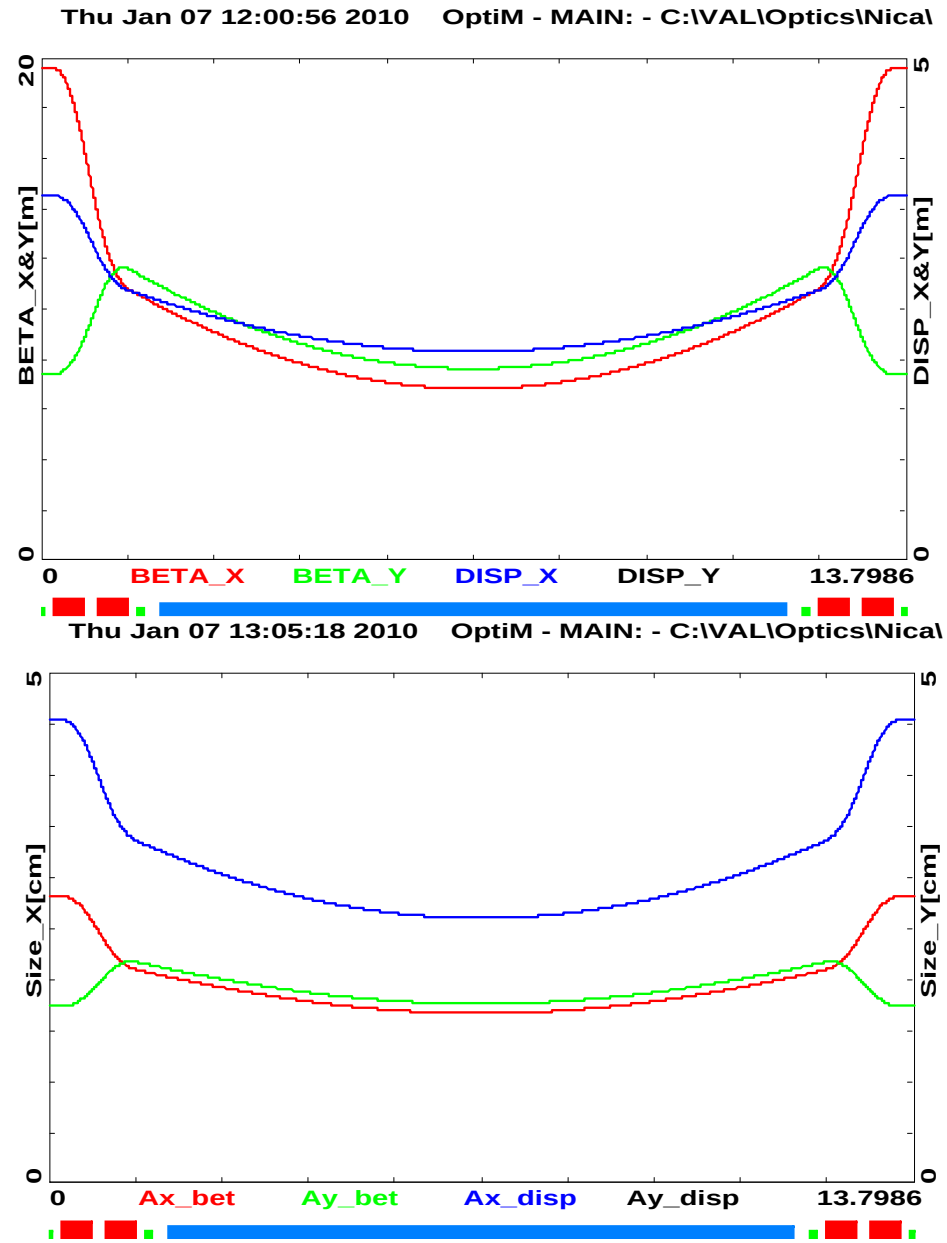
Beam envelopes for ε = 40 mm mrad

Separation length - 10.3 m,

Distance to first parasitic collision – 11.35 m

Periodicity Element

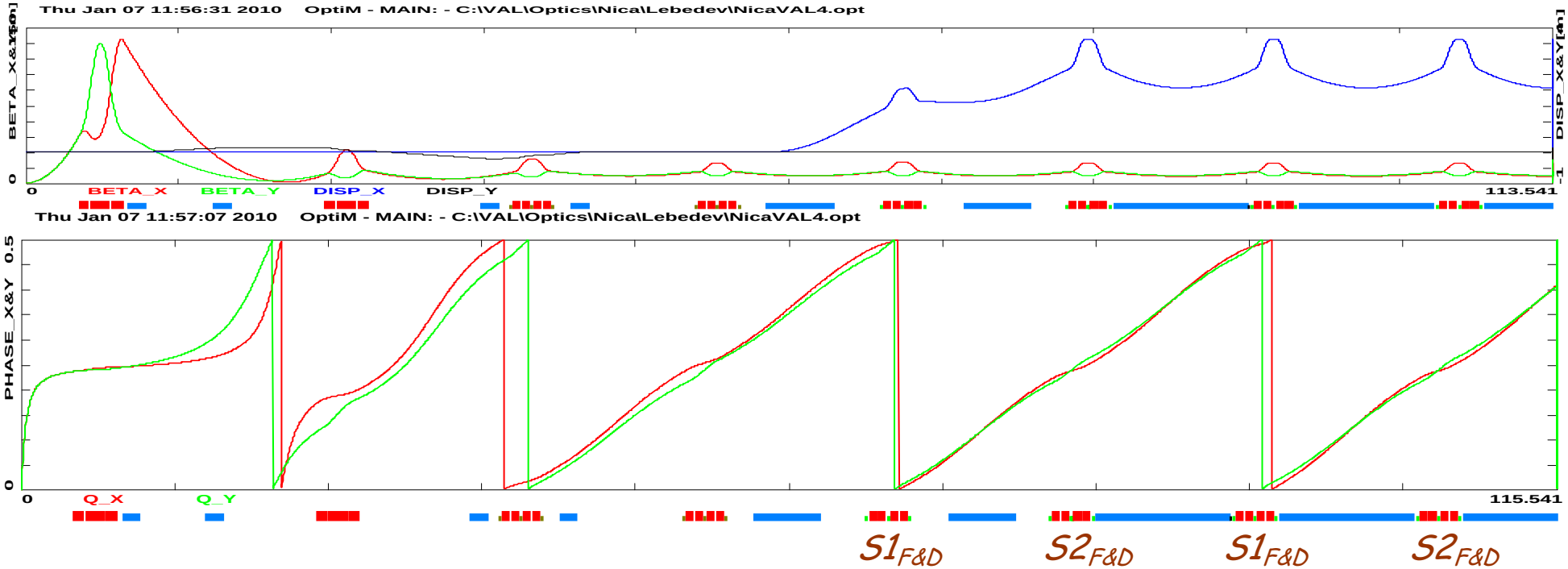
N	Name	S [cm]	L [cm]	B [kG]	G [kG/cm]	S [kG/cm^2]
1	sFh	5	5			0.067
2	op3	20	15			
3	qF	70	50		3.2639	
4	op2	90	20			
5	qD	140	50		-3.4398	
6	op3	155	15			
7	sD1	165	10			-0.103
8	op1	190	25			
10	bH	1189.86	999.861	20		
12	op1	1214.86	25			
13	sD2	1224.86	10			
14	op3	1239.86	15			
15	qD	1289.86	50		-3.4398	
16	op2	1309.86	20			
17	qF	1359.86	50		3.2639	
18	op3	1374.86	15			
19	sFh	1379.86	5			0.067



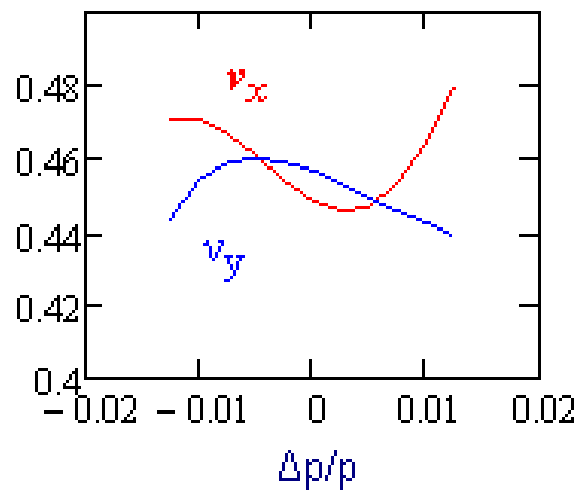
ε = 40 mm mrad, $\Delta p/p$ = 0.0125%

- Central quad is split to insert S_F

Chromaticities of tunes and β-functions

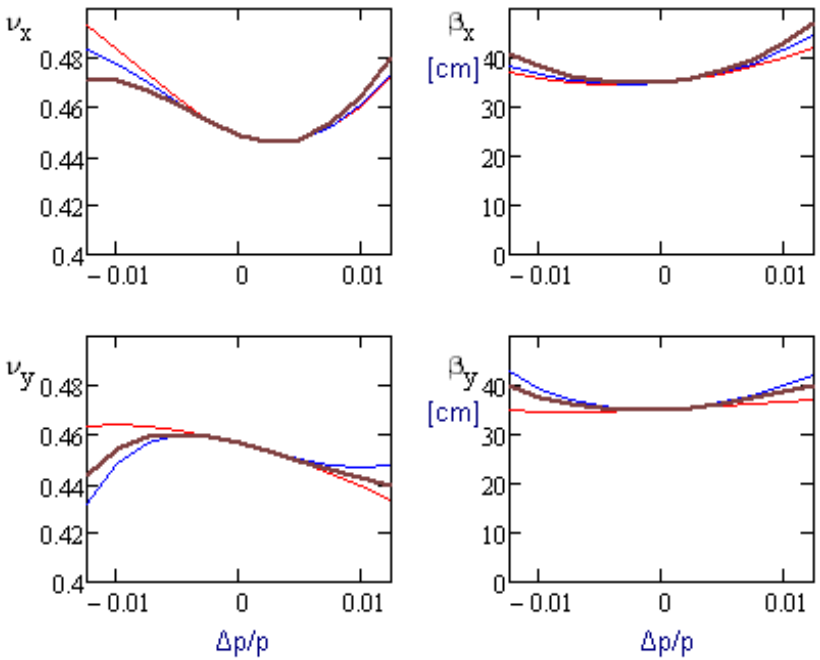


- Tune dependence on momentum is very nonlinear even for ideal dipoles and quads
- Sextupoles were adjusted to minimize tune variations
- Natural chromaticities:
  ξ_x = -27.1, ξ_y = -23.2 ($\Delta\xi_{xy}$~-17 from 2 IPs)
- Corrected chromaticities:
  ξ_x = -1.54, ξ_y = -1.50



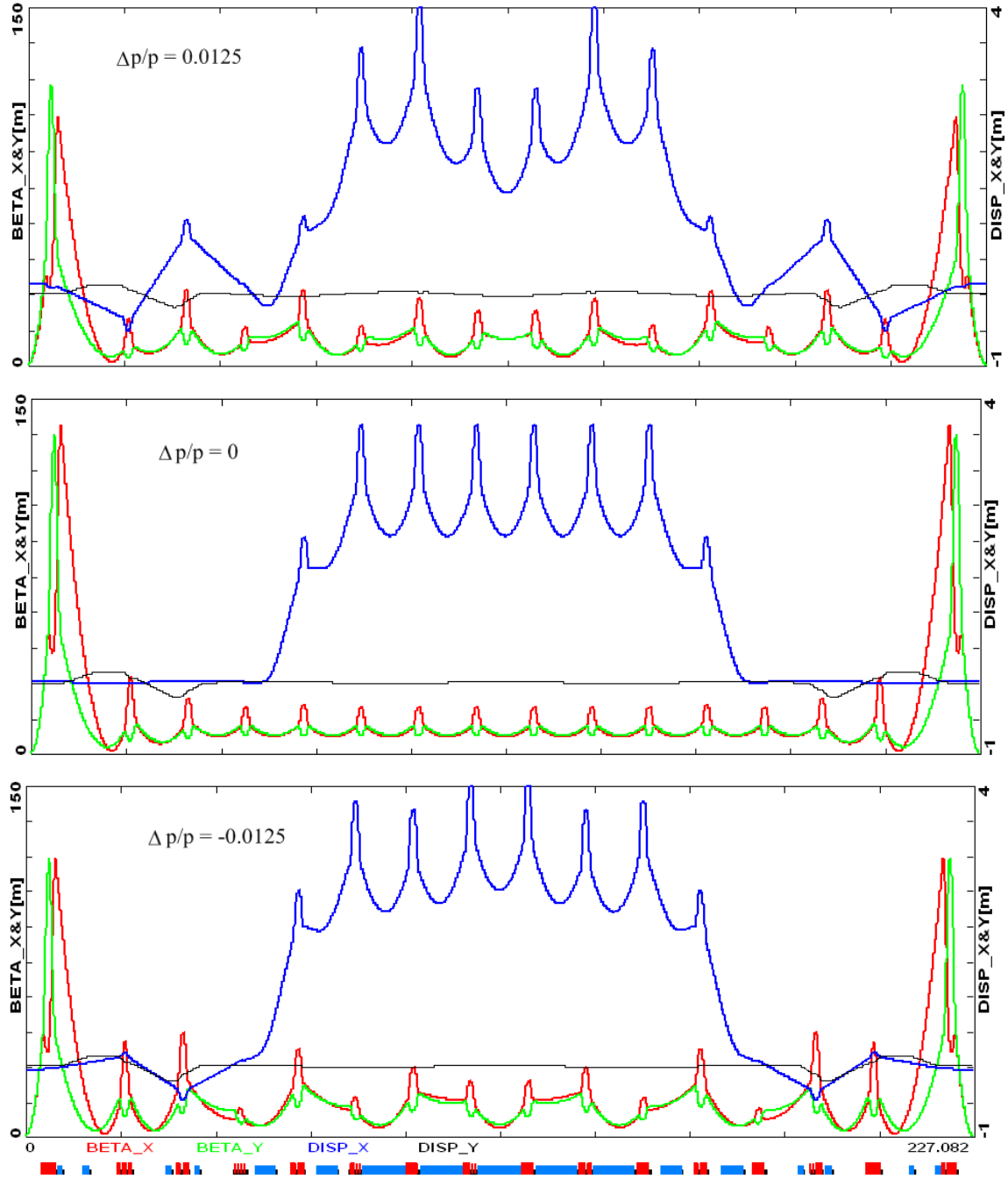
	S_F[kG/cm^2]	S_D[kG/cm^2]
S_1	0.386	-0.290
S_2	0.386	-0.336

Dependence of Twiss Functions on Momentum

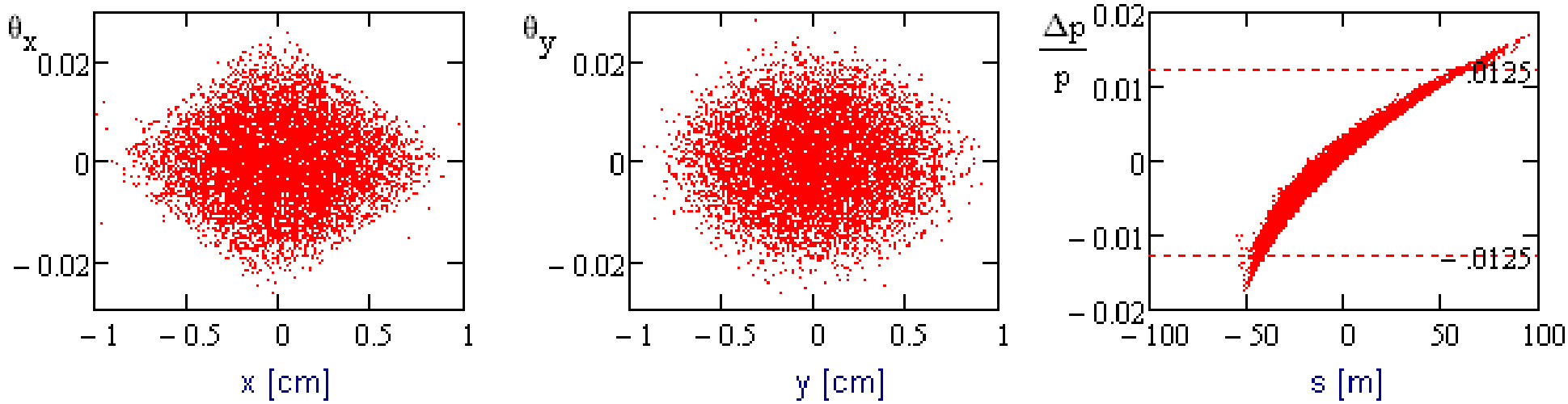


	S_F [kG/cm^2]	S_D [kG/cm^2]
S_1	0.386	-0.316
S_2		
S_1	0.274	-0.188
S_2	0.473	-0.416
S_1	0.386	-0.290
S_2	0.386	-0.336

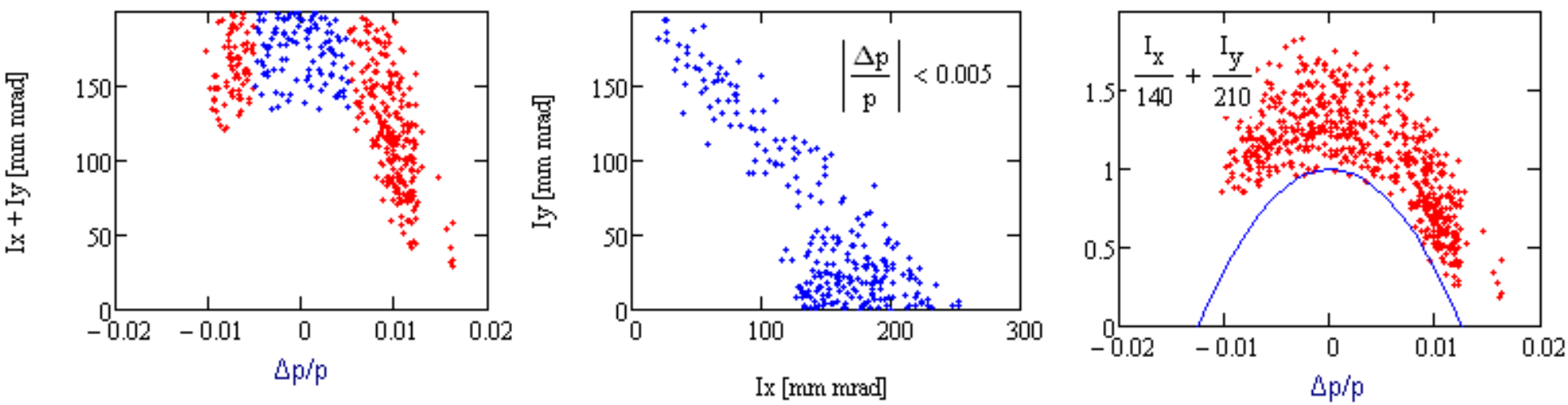
- Correction of β-function worsens tune correction



Dynamic Aperture (no particle interaction)



Phase space projections in IP for particles survived after 5000 turns (no RF)



Actions (C-S inv.) in IP for initial coordinates of particles lost after 5000 turns, ε_{xmax}=140 mm mrad, ε_{ymax} = 210 mm mrad, $(Dp/p)_{max}$=0.0125

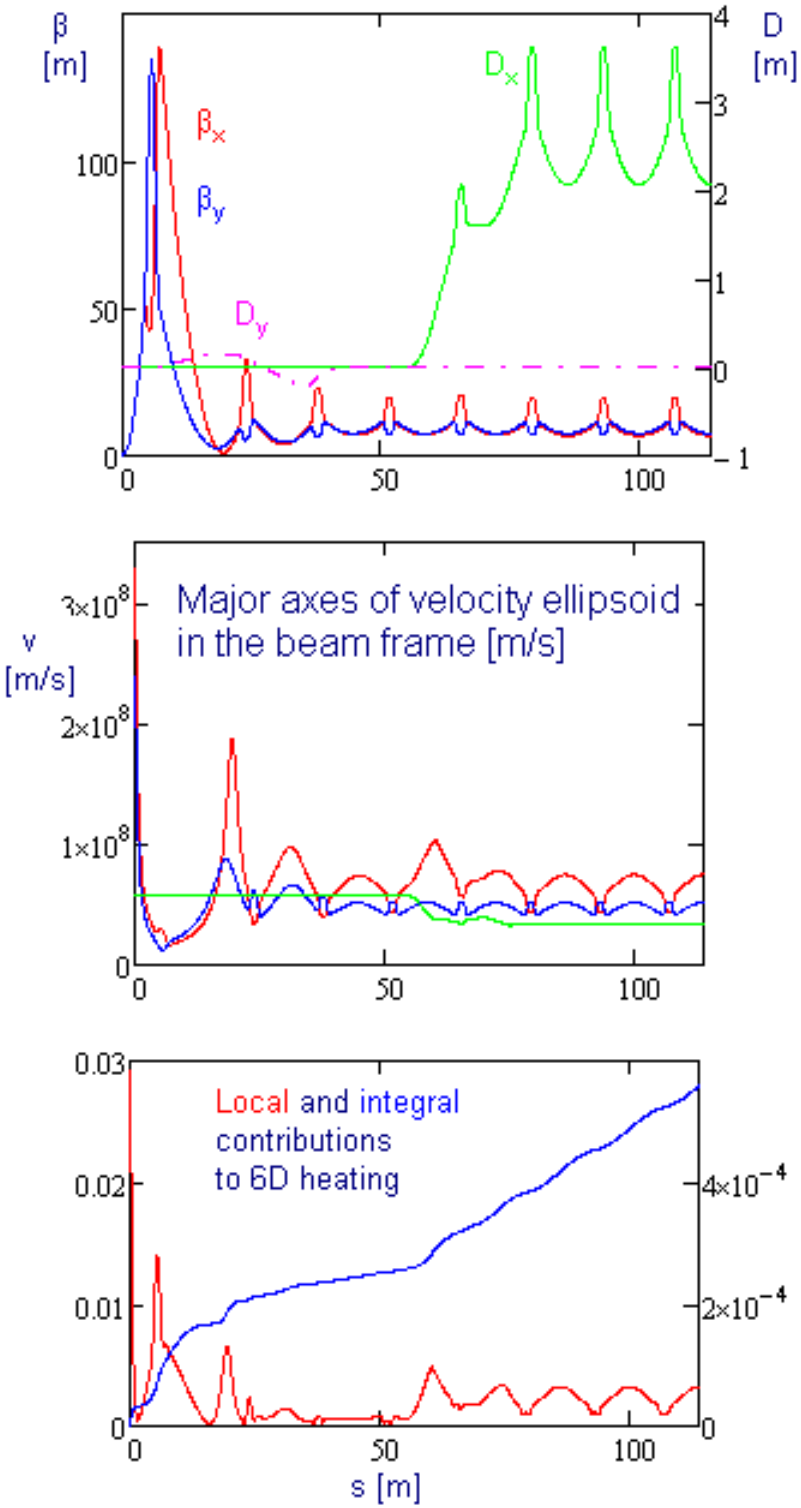
$$\frac{I_x}{\varepsilon_{x_{\max}}} + \frac{I_y}{\varepsilon_{y_{\max}}} + \left(\frac{\Delta p/p}{(\Delta p/p)_{\max}}\right)^2 < 1$$

- Dynamic aperture exceeds the requirement of 40 mm mrad by ~2 times

IBS in Collider

- Straight lines and IPs increase IBS heating by about 4.5 times
  - However operation in vicinity of thermal equilibrium still significantly reduces IBS heating
    - 3.6 times difference between heating of all degrees of freedom and temperature exchange

	ε_x [mm mrad]	ε_y [mm mrad]	σ_p %	$\tau_x=\tau_y=\tau_s$ [s]	$\tau_{exchange}$ [s]
3.5 GeV/n	1.117	0.692	0.156	1025	227
4.5 GeV/n	1.291	0.684	0.192	1350	374



Stochastic Cooling

- Slip-factor was chosen to optimize stochastic cooling at 4.5 GeV/n
  - Band 3-6 GHz
- Combined system for transverse and longitudinal cooling can save space and money
  - Hybrids form Δ & Σ from pickup signals
  - Separate amplification and signal treatment circuits
  - Hybrids form Δ & Σ signals for kickers (same kicker for L & (X or Y))
- Filter cooling for L cooling
- Other sign partial slip-factor (pickup-to-kicker) compensates the bad mixing of filter cooling

Theory of Filter Stochastic cooling

$$\frac{\partial\psi(x,t)}{\partial t}+\frac{\partial}{\partial x}\left(F(x)\psi(x,t)\right)=\frac{1}{2}\frac{\partial}{\partial x}\left(D(x)\frac{\partial\psi(x,t)}{\partial x}\right)$$

$$F(x)=\frac{1}{T_0}\sum_{n=-\infty}^{\infty}\frac{G\left(x,\omega_n(x)\right)}{\varepsilon\left(\omega_n(x)\right)}\left(1-A\left(\omega_n(x)\right)e^{-i\omega_n(x)T_0}\right)e^{i\omega_n(x)\left(T_2(x)-T_{20}\right)}\xrightarrow[\substack{\omega_n=n\omega_0(1-\eta x)\\ T_2(x)-T_{20}=T_0\eta_2 x}]{\substack{Neglect\ \varepsilon\\ Perfect\ NF}}$$

$$\frac{1}{T_0}\sum_{n=-\infty}^{\infty}G_n\left(1-e^{2\pi in\eta x}\right)e^{2\pi in\eta_2 x}\xrightarrow{G_n=-iG_0\ \ for\ n\subset[n_1,n_2]}-\frac{4G_0}{T_0}\sum_{n=n_1}^{n_2}\sin\left(\pi n\eta x\right)\cos(\pi n(\eta+2\eta_2)x)$$

$$\approx-\frac{4G_0}{T_0}\left(n_2-n_1\right)\sin\left(\pi\frac{n_1+n_2}{2}\eta x\right)\cos\left(\pi\frac{n_1+n_2}{2}(\eta+2\eta_2)x\right)\xrightarrow[\pi n_2\eta x<<1]{\pi n\eta x<<1}-\frac{2\pi G_0}{T_0}\left(n_2^{\ 2}-n_1^{\ 2}\right)\eta x$$

$$D(x)=\frac{N}{T_0}\psi(x)\sum_{n=-\infty}^{\infty}\frac{1}{\left|\varepsilon\left(\omega_n(x)\right)^2\right|\left|n\eta\right|}\left|G(x,\omega_n(x))\left(1-A\left(\omega_n(x)\right)e^{-i\omega_n(x)T_0}\right)\right|^2\xrightarrow[\omega_n=n\omega_0(1-\eta x)]{\substack{Neglect\ \varepsilon\\ Perfect\ NF}}$$

$$\frac{N}{T_0\left|\eta\right|}\psi(x)\sum_{n=-\infty}^{\infty}\frac{1}{\left|n\right|}\left|G_n\left(1-e^{2\pi in\eta x}\right)\right|^2\xrightarrow{G_n=-iG_0\ \ for\ n\subset[n_1,n_2]}\frac{8NG_0^{\ 2}}{T_0\left|\eta\right|}\psi(x)\sum_{n=n_1}^{n_2}\frac{\sin(\pi n\eta x)^2}{n}$$

$$\xrightarrow{\pi n\eta x<<1}\frac{4\pi^2\left|\eta\right|NG_0^{\ 2}}{T_0}\left(n_2^{\ 2}-n_1^{\ 2}\right)\psi(x)x^2$$

Finally for small x one obtains:

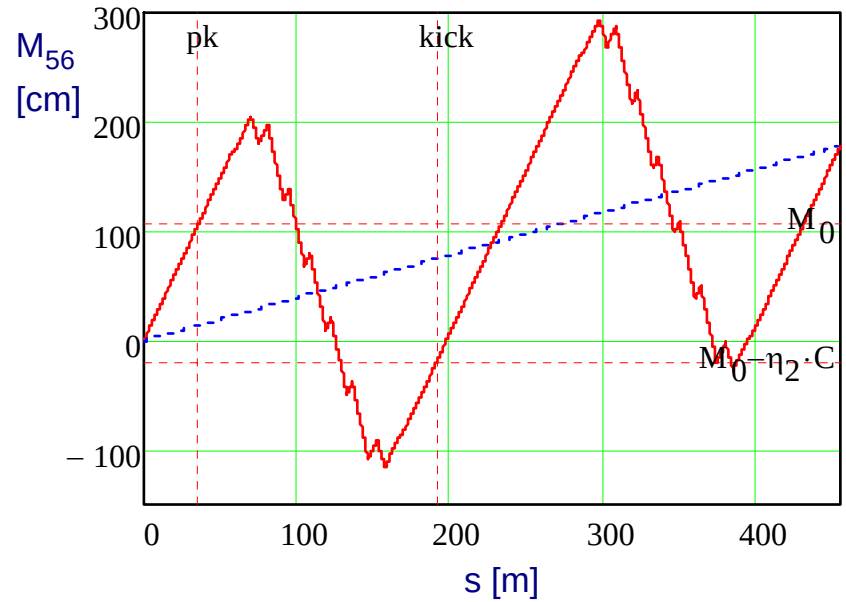
$$\boxed{F(x)=-G_0F_x x,\quad D(x)=G_0^{\ 2}D_{xx}\psi(x)x^2,\quad F_x=\frac{2\pi\eta}{T_0}\left(n_2^{\ 2}-n_1^{\ 2}\right),\quad D_{xx}=\frac{4\pi^2\left|\eta\right|N}{T_0}\left(n_2^{\ 2}-n_1^{\ 2}\right)}$$

Bad Mixing for Filter Cooling

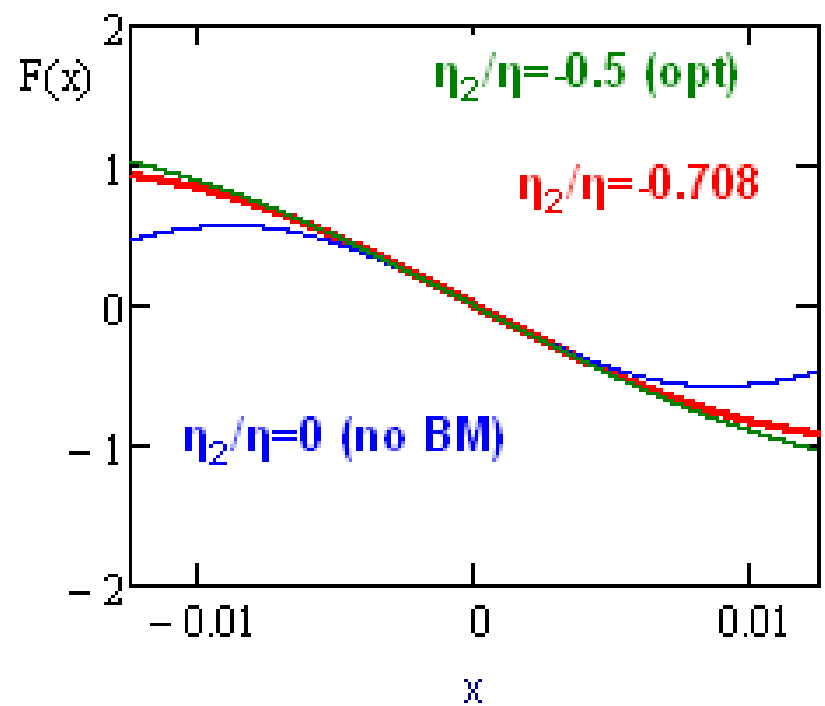
$$F(x) = -\frac{4G_0}{T_0}\sum_{n=n_1}^{n_2}\sin(\pi n\eta x)\cos(\pi n(\eta+\eta_2)x)$$

- $\eta_2 = -\eta/2$ maximizes width of the cooling force

$$F(x) = -\frac{4G_0}{T_0}\sum_{n=n_1}^{n_2}\begin{cases}\sin(2\pi n\eta x)/2, & \text{for } \eta_2 = -\eta/2\\ \sin(\pi n\eta x), & \text{for } \eta_2 = 0\end{cases}$$



- If optics is unchanged the slip factors will grow fast with energy decrease
  - Minor decrease will make η_2 optimal



Optimal Gain of Filter Cooling

$$\int x^2 dx \left\{ \frac{\partial \psi(x,t)}{\partial t} + \frac{\partial}{\partial x}\left(F(x)\psi(x,t)\right) = \frac{1}{2}\frac{\partial}{\partial x}\left(D(x)\frac{\partial \psi(x,t)}{\partial x} \right) \right\}$$

$$\frac{d}{dt}\overline{x^2} = 2\int xF(x) f_0(x,t) + \int f_0(x,t)\frac{\partial}{\partial x}\left(xD(x)\right)dx$$

- Substituting $F(x) = -G_0 F_x x, \quad D(x) = G_0{}^2 D_{xx}\psi(x)x^2$ one obtains

$$\frac{d}{dt}\overline{x^2} = -2G_0 F_x \overline{x^2} + G_0{}^2 D_{xx}\int \psi(x)\frac{\partial}{\partial x}\left(x^3\psi(x)\right)dx \xrightarrow[\overline{x^2}=\sigma]{\text{for Gaussian distribution}} -2F_x G_0 \sigma^2 + \frac{3D_{xx}}{8\sqrt{\pi}} G_0{}^2 \sigma$$

Optimal gain is $G_{opt} = \dfrac{8\sqrt{\pi}F_x\sigma}{3D_{xx}}$

And cooling decrement at optimal gain is $\lambda\left(G_{opt}\right) = \dfrac{8\sqrt{\pi}F_x{}^2\sigma}{3D_{xx}}$

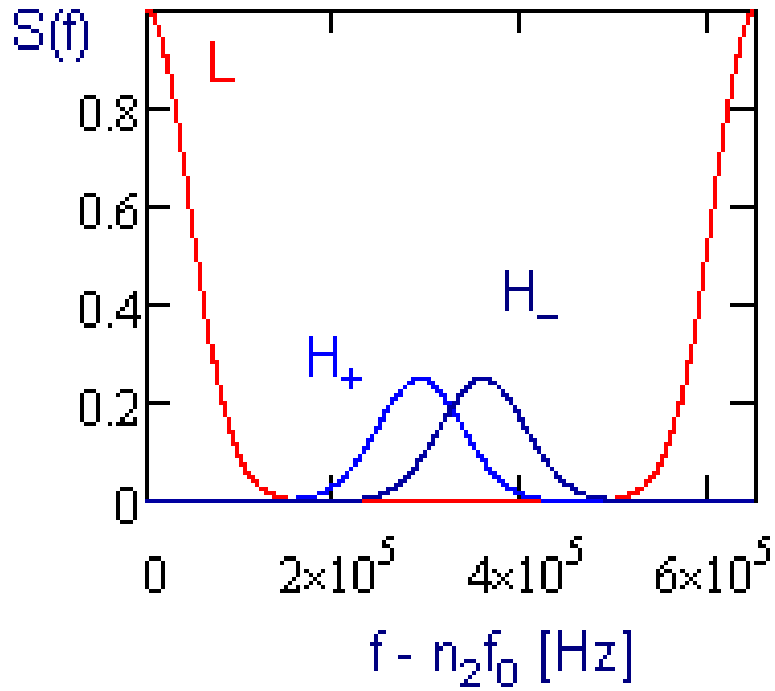
- In practical terms: $\boxed{\lambda\left(G_{opt}\right) = \dfrac{8\sqrt{\pi}\sigma_p\eta}{3T_0 N}\left(n_2{}^2 - n_1{}^2\right)\dfrac{\sqrt{2\pi}\sigma_s n_b}{C}}$

  The last term takes into account that the beam is bunched

  - For NICA at 4.5 GeV and 3-6 GHz cooling system one obtains the cooling time of 63 s

Band overlap

- It is highly undesirable to have an overlap of transverse and longitudinal spectra
  - That sets maximum frequency to 6 GHz
- As one can see that the stochastic cooling in the optimal case is about an order of magnitude faster than the IBS growth rate in the state of "thermal equilibrium
  - It allows one significant deviation of beam emittances from the "equilibrium" if necessary
  - To avoid significant degradation of stochastic cooling at 3.5 GeV one needs to change the slip factor and, consequently, the beam optics
- Expected emittance growth time due to scattering on residual gas is expected to be more than 1 hour and should not be a problem



RF and Microwave Instability

	3.5 GeV/n	4.5 GeV/n
RF frequency (fundamental)	25.81	26.01
Harmonic number, h	40	
Slip-factor	-0.01885	-0.003926
RF voltage, kV	760	430
Bunch length, σ_s, cm	72	60
Momentum spread, σ_p	$1.56 \cdot 10^{-3}$	$1.92 \cdot 10^{-3}$
Zn/n (mainly space charge)	24 i	15i
Zn/n at stability boundary(circle)	342	70

- Due to operation below transition and large momentum spread there is a reserve for microwave instability of more than an order of magnitude
- 3 RF systems are required
  - (1) fundamental (h=40); (2) bunching (h=20); (3) injection (barrier bucket
- If larger bunch length is acceptable from the particle detectors point of view the h=40 RF system can be eliminated.
  - That will also result in a luminosity increase (for fixed $\delta\nu_{SC}$)

Vacuum

- Good vacuum is highly desired to achieve good beam lifetime
  - 10^9 Torr H_2 equivalent results in
    - ~300 hour beam lifetime with approximately equal contributions from electromagnetic and nuclear scattering
    - ~5 hour emittance growth time due to multiple scattering on the residual gas

Backup transparencies

IBS

- For plasma

$$\frac{d}{dt}\begin{pmatrix}\sigma_{\mathrm{vx}}{}^2\\ \sigma_{\mathrm{vy}}{}^2\\ \sigma_{\mathrm{vz}}{}^2\end{pmatrix}=\frac{(2\pi)^{3/2}\,nr_0{}^2c^4L_c}{\sqrt{\sigma_{\mathrm{vx}}^2+\sigma_{\mathrm{vy}}^2+\sigma_{\mathrm{vz}}^2}}\begin{pmatrix}\Psi\left(\sigma_{\mathrm{vx}},\sigma_{\mathrm{vy}},\sigma_{\mathrm{vz}}\right)\\ \Psi\left(\sigma_{\mathrm{vy}},\sigma_{\mathrm{vz}},\sigma_{\mathrm{vx}}\right)\\ \Psi\left(\sigma_{\mathrm{vz}},\sigma_{\mathrm{vx}},\sigma_{\mathrm{vy}}\right)\end{pmatrix}$$

$$\rho_{\min}=r_0c^2/\overline{\mathrm{v}^2}\,,$$

$$\rho_{\max}=\sqrt{\overline{\mathrm{v}^2}/4\pi nr_0c^2}\,,$$

$$\overline{\mathrm{v}^2}=\sigma_{\mathrm{vx}}^2+\sigma_{\mathrm{vy}}^2+\sigma_{\mathrm{vz}}^2\,.$$

$$\Psi(x,y,z)=\frac{\sqrt{2}r}{3\pi}\Big(y^2\,\mathrm{R_D}\left(z^2,x^2,y^2\right)$$
$$+z^2\,\mathrm{R_D}\left(x^2,y^2,z^2\right)-2x^2\,\mathrm{R_D}\left(y^2,z^2,x^2\right)\Big),$$

$$\mathrm{R_D}(x,y,z)=\frac{3}{2}\int_0^\infty\frac{dt}{\sqrt{(t+x)(t+y)(t+z)^3}}$$

- For beam

In the absence of *x-y* coupling for a bunched beam non-relativistic in the beam frame the emittance growth rates can be expressed in a compact matrix form:

$$\frac{d\varepsilon_k}{dt}=\frac{Nr_0{}^2c^2}{4\sqrt{2}\sigma_z\beta^2\gamma^4\sqrt{\varepsilon_x\varepsilon_y}}\left\langle\frac{L_c\sum_{i,j=1}^{3}\mathbf{B}_{ij}^k\mathbf{R}_{ij}}{\sqrt{\beta_x\beta_yF_D\,\mathrm{tr}(\boldsymbol{\Sigma})}}\right\rangle_s,$$

Here $\langle ... \rangle_s$ denotes averaging over the machine circumference, β and γ are the relativistic factors; N is the number of particles per bunch; $\mathbf{\Sigma} \equiv \left[\overline{\mathrm{v}_i \mathrm{v}_j} \right]$ is the matrix of the second moments of local velocity distribution in the BF,

$$\mathbf{\Sigma} = \left(\gamma\beta c\right)^2 \mathbf{G}^T \mathbf{\Xi}^{-1} \mathbf{G}$$

$\mathbf{\Xi}$ is the bilinear form of particle angles ($\theta_x, \theta_y, \Delta p/p$),

$$\mathbf{\Xi} = \begin{pmatrix} \beta_x / \varepsilon_x & 0 & -\beta_x \Phi_x / \varepsilon_x \\ 0 & \beta_y / \varepsilon_y & -\beta_y \Phi_y / \varepsilon_y \\ -\beta_x \Phi_x / \varepsilon_x & -\beta_y \Phi_y / \varepsilon_y & \mathbf{\Xi}_{33} \end{pmatrix},$$

$$\mathbf{\Xi}_{33} = 1/\sigma_p{}^2 + A_x / \varepsilon_x + A_y / \varepsilon_y \; ;$$

$$\Phi_x = D'_x + \alpha_x D_x / \beta_x \,, \quad \Phi_y = D'_y + \alpha_y D_y / \beta_y \,; \; A_x = \left(D_x{}^2 + \left(\beta_x \Phi_x\right)^2 \right) / \beta_x \,, A_y = \left(D_y{}^2 + \left(\beta_y \Phi_y\right)^2 \right) / \beta_y$$

$$F_D = 1 + D_x{}^2 \sigma_p{}^2 / \left(\varepsilon_x \beta_x\right) + D_y{}^2 \sigma_p{}^2 / \left(\varepsilon_y \beta_y\right) \,;$$

$\sigma_z{}^2$ is the squared rms bunch length; β_x, β_y, α_x, and α_y, are the beta-functions and their negative half derivatives; D_x, D_y, D'_x and D'_y are the dispersions and their derivatives; ε_x, ε_y and $\varepsilon_z = \sigma_z \sigma_p$ are the unnormalized transverse and longitudinal rms beam emittances; and σ_p is the relative rms momentum spread; matrices $\mathbf{B}^k$ and $\mathbf{G}$ are:

$$\mathbf{B}^x = \begin{pmatrix} \beta_x & 0 & \Phi_x\beta_x \\ 0 & 0 & 0 \\ \Phi_x\beta_x & 0 & A_x \end{pmatrix}, \mathbf{B}^y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta_y & \Phi_y\beta_y \\ 0 & \Phi_y\beta_y & A_y \end{pmatrix},$$

$$\mathbf{B}^z = \mathrm{diag}\left(0, 0, \beta_z\right), \quad \mathbf{G} = \mathrm{diag}\left(1, 1, 1/\gamma\right),$$

the function diag(..) makes a diagonal matrix from a vector; $\beta_z = \sigma_z/\sigma_p$ is the longitudinal beta-function;

$$\mathbf{R} = \left(\mathbf{G}^{-1}\right)^T \mathbf{T}\mathbf{\Psi}_{IBS}\left(\mathbf{T}^T\mathbf{\Sigma}\mathbf{T}\right)\mathbf{T}^T\mathbf{G}^{-1};$$

a rotational matrix **T** reduces Σ to its diagonal form $\boldsymbol{\sigma} = \mathbf{T}^T\mathbf{\Sigma}\mathbf{T}$; and the matrix function $\mathbf{\Psi}_{IBS}(\boldsymbol{\sigma})$ with its argument of a diagonal matrix σ is:

$$\mathbf{\Psi}_{IBS}\left(\boldsymbol{\sigma}\right) = \mathrm{diag}\Big(\Psi\left(\sigma_{11}, \sigma_{22}, \sigma_{33}\right),$$
$$\Psi\left(\sigma_{22}, \sigma_{33}, \sigma_{11}\right), \Psi\left(\sigma_{33}, \sigma_{11}, \sigma_{22}\right)\Big).$$

The Coulomb logarithm is computed similarly to the plasma case with the following correction affecting the value of maximum impact parameter:

$$\rho_{\max} = \min\left(\sigma_{\min}, \gamma\sigma_z, \sqrt{\mathrm{tr}\left(\mathbf{\Sigma}\right)/4\pi n r_0 c^2}\right),$$

$$2\sigma_{\min}{}^2 = \sigma_x{}^2 + \sigma_y{}^2 - \sqrt{\left(\sigma_x{}^2 - \sigma_y{}^2\right)^2 + 4D_x{}^2 D_y{}^2 \sigma_p{}^4}\ ,$$

where

$$\sigma_x{}^2 = \varepsilon_x \beta_x + D_x{}^2 \sigma_p{}^2, \quad \sigma_y{}^2 = \varepsilon_y \beta_y + D_y{}^2 \sigma_p{}^2\ .$$

The above equations can be used for a coasting beam with following substitutions: $\sigma_z \to L/\left(2\sqrt{\pi}\right)$, $\mathbf{B}_z = \mathrm{diag}\left(0,0,2\right)$ implying that $d\varepsilon_3/dt \to d\overline{\sigma_p{}^2}/dt$. Note that the factor of 2 in B_z reflects the absence of the synchrotron motion, taken into account in the above equation.